JOURNAL OF DIFFERENTIAL EQUATIONS **139**, 146–193 (1997)
ARTICLE NO. DE973285

Unfolding of a Quadratic Integrable System with Two Centers and Two Unbounded Heteroclinic Loops

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Received March 4, 1997

In this paper we present a complete study of quadratic 3-parameter unfoldings of some integrable system belonging to the class Q_3^R, and having two centers and two unbounded heteroclinic loops. We restrict to unfoldings that are transverse to Q_3^R, obtain a versal bifurcation diagram and all global phase portraits, including the precise number and configuration of the limit cycles. It is proved that 3 is the maximal number of limit cycles surrounding a single focus, and only the (1, 1)-configuration can occur in case of simultaneous nests of limit cycles. Essentially the proof relies on a careful analysis of a related non-conservative Abelian integral. © 1997 Academic Press

1. INTRODUCTION AND STATEMENT OF THE RESULTS

Essential in many bifurcation problems is the study of the limit cycles occuring in perturbations of Hamiltonian systems

$$\begin{cases} \dot{x} = -\dfrac{\partial H}{\partial y} + \varepsilon f, \\ \dot{y} = \dfrac{\partial H}{\partial x} + \varepsilon g. \end{cases} \tag*{$(1.1)_\varepsilon$}$$

* Partially supported by NSF and DEPF of China.

0022-0396/97 $25.00

In first order approximation, with respect to ε, this problem requires studying the number of zeros for the function, defined by the Abelian integral

$$I(h)=\int_{H=h} f\,dy-g\,dx, \tag{1.2}$$

where $h\in(h_1, h_2)$, and for each h, the set $\Gamma_h=\{(x, y)\,|\,H(x, y)=h\}$ is compact.

In fact, the weak Hilbert's 16-th problem, proposed by V.I. Arnold [A], is asking for an upper bound for the number of zeros of $I(h)$, where $H(x, y)$ is a polynomial of degree $n+1$, $f(x, y)$ and $g(x, y)$ are polynomials of degree $\leqslant n$.

DEFINITION 1.1 [Ma]. *The perturbation in* $(1.1)_\varepsilon$ *is called non-conservative if* $I(h)\not\equiv 0$ *for* $h\in(h_1, h_2)$, *where* $I(h)$ *is defined in* (1.2).

Hence, if the perturbation is non-conservative and Γ_h is connected and does not contain any singularity of $(1.1)_0$ for $h\in[h_1, h_2]$, then the number of limit cycles of $(1.1)_\varepsilon$, for ε small, is smaller than or equal to the number of zeros of $I(h)$, taking into account the multiplicity.

If, however, Γ_{h^*} contains some singularities of $(1.1)_0$ for a $h^*\in[h_1, h_2]$, then one has to consider the problem for h near h^* by other tools, such as the theory of Hopf bifurcations, and of homoclinic (or heteroclinic) bifurcations, depending on the nature of the singularities.

The two problems mentioned above are not solved completely even not for the polynomial case $n=2$. In the quadratic case it is naturally to consider perturbations from centers, as has been done by H. Żołądek in [Z1]. He divided the quadratic centers into four classes : Q_3^{LV}, Q_3^H, Q_3^R and Q_4, called Lotka–Volterra case, Hamiltonian case, reversible case and codimension 4 case respectively. For each case he proposed a conjecture about the maximal number of limit cycles to be encountered after perturbation. In general, concerning the number of limit cycles it is more difficult to find an upperbound than a lowerbound and it is quite hard to find the precise upperbound and the configuration of limit cycles surrounding each of the respective singularities. It should be noticed here that if the unperturbed system belongs to $Q_3^{LV}\cup Q_3^R\cup Q_4\backslash Q_3^H$, one has to multiply it by an integrating factor to put the perturbed system in the form $(1.1)_\varepsilon$. But H, f and g are then no longer polynomials; this may cause some new difficulties. We however still call (1.2) an Abelian integral, and accordingly quote the corresponding question, looking for an upper bound of the number of zeros of $I(h)$ as weak Hilbert 16th-problem.

Let us list here some results concerning the above mentioned questions for $n=2$. If $(1.1)_0\in Q_3^H$, then two is the maximal number of limit cycles that can bifurcate from any homoclinic loop, as proved in [HI1] and [I]; and two is the maximal number of limit cycles for $(1.1)_\varepsilon$ if, in addition, $(1,1)_0$ has three saddle points and one center as proved in [HI2]. If the system has a separatrix cycle in the form of a bounded triangle (resp. unbounded triangle or conic segment), then it belongs to Q_3^{LV}, (resp. $Q_3^{LV} \cap Q_3^R \backslash Q_3^H$ or $Q_3^{LV} \cap Q_3^R \cap Q_3^H$), and three (resp. two) is the maximal number of limit cycles near this configuration after perturbation as proved in [Z2]. If the unperturbed system is non-Hamiltonian and has a homoclinic loop, then it must belong to Q_3^R, and two is the maximal number of limit cycles that can bifurcate from the loop, if the perturbations are non-conservative (except for one case), as proved in [HL]. In [SZ] a subset of Q_3^R is given consisting of systems from which at least three limit cycles can appear by perturbation. Besides, in this paper some bifurcation diagrams are conjectured.

H. Żołądek pointed out in [Z1] that the stratum Q_3^R is the most complicated one, and the important problem is to check which configurations of limit cycles are possible. Any $X_0 \in Q_3^R$ can be transformed into the form

$$\begin{cases} \dot{x} = -y + ax^2 + by^2, \\ \dot{y} = x(1+cy), \end{cases} \tag{1.3}$$

where (without loss of generality) $c<0$. If the parameters satisfy

$$a<c<0<b<-c, \tag{1.4}$$

then the phase portrait of X_0 in the Poincaré sphere is shown in Fig. 1. There are two centers in the finite plane, and three (pairs of) singularities at infinity: two saddles and one node. Each of the two center regions is surrounded by an unbounded heteroclinic loop, which consists of the heteroclinic orbit, joining two saddles at infinity, and a part of the equator at infinity in the Poincaré disc.

In the present paper we take one of the systems (1.3) under condition (1.4), and make a complete study of the 3-parameter unfoldings which are transverse to the stratum Q_3^R. The related problem on Abelian integrals will

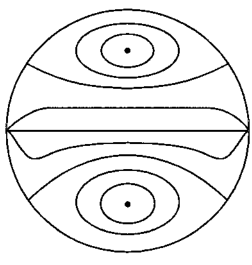

FIG. 1. Phase portrait of organizing center.

reveal to be non-conservative. In fact, we will essentially consider the family of systems

$$\begin{cases} \dot{x} = -y - 3x^2 + y^2 + \mu_1 x + \mu_2 xy, \\ \dot{y} = x(1-2y) + \mu_3 x^2, \end{cases} \tag{1.5$_\mu$}$$

where $\mu = (\mu_1, \mu_2, \mu_2) \in \mathbb{R}^3$ small.

The main result is the following

THEOREM 1.2. (i) *The* 3-*parameter family* $(1.5)_\mu$ *is a versal unfolding of* $(1.5)_0$, *among all* 3-*parameter unfoldings of* $(1.5)_0$, *transverse to the stratum* Q_3^R.

(ii) *The bifurcation diagram of* $(1.5)_\mu$ *has a conic structure in* $\mathbb{R}^3$ *for* $0 < |\mu| \ll 1$, *and it is point-symmetric with respect to* $\mu = 0$. *Hence it can be expressed* (*for* $\mu \neq 0$) *by drawing its intersection with the half sphere* $S_\gamma^+ : \{\mu_1^2 + \mu_2^2 + \mu_3^2 = r^2,\ \mu_3 \geqslant 0 \mid r > 0 \text{ small}\}$.

(iii) *The intersection of the bifurcation diagram with* S_r^+ (*projected into* $\mathbb{R}^2$), *and the related structurally stable phase portraits are as shown in Fig.* 2.

In Fig. 2, the notations H, DH (resp. L, DL) indicate the Hopf (resp. homoclinic loop) bifurcation of order 1 and two; DC and TC denote respectively double and triple limit cycle bifurcations, and so on. The subscripts 1 and 2 refer to the respective singularities at which or around which the bifurcations happen.

Remark 1.3. Seemingly it is very difficult to make a complete study of the vector fields near a general $X \in Q_3^R$; for that we restrict our attention to a specific $X_0 \in Q_3^R$. Our interest in this case relies on following facts:

A complete treatment of the total number and the configuration of limit cycles, surrounding each of the two foci, by application of a geometric method, with potentiality for use in other situations, for proving the occurence of at most two or at most three limit cycles.

The treatment permits to present a method to study bifurcations of unbounded heteroclinic loops. It relies on the Poincaré transformation bringing the unbounded loops into a compact region, such that the perturbation theory and the implicit function theorem can be used. In the meantime, some interesting configurations are found: two limit cycles can be surrounded by such a loop, and two such loops can co-exist.

The paper is organized as follows. In Section 2, we give some preliminary results concerning the conclusions (i) and (ii) of Theorem 1.2. We introduce an Abelian integral $I(h)$ and reduce it to an equivalent form

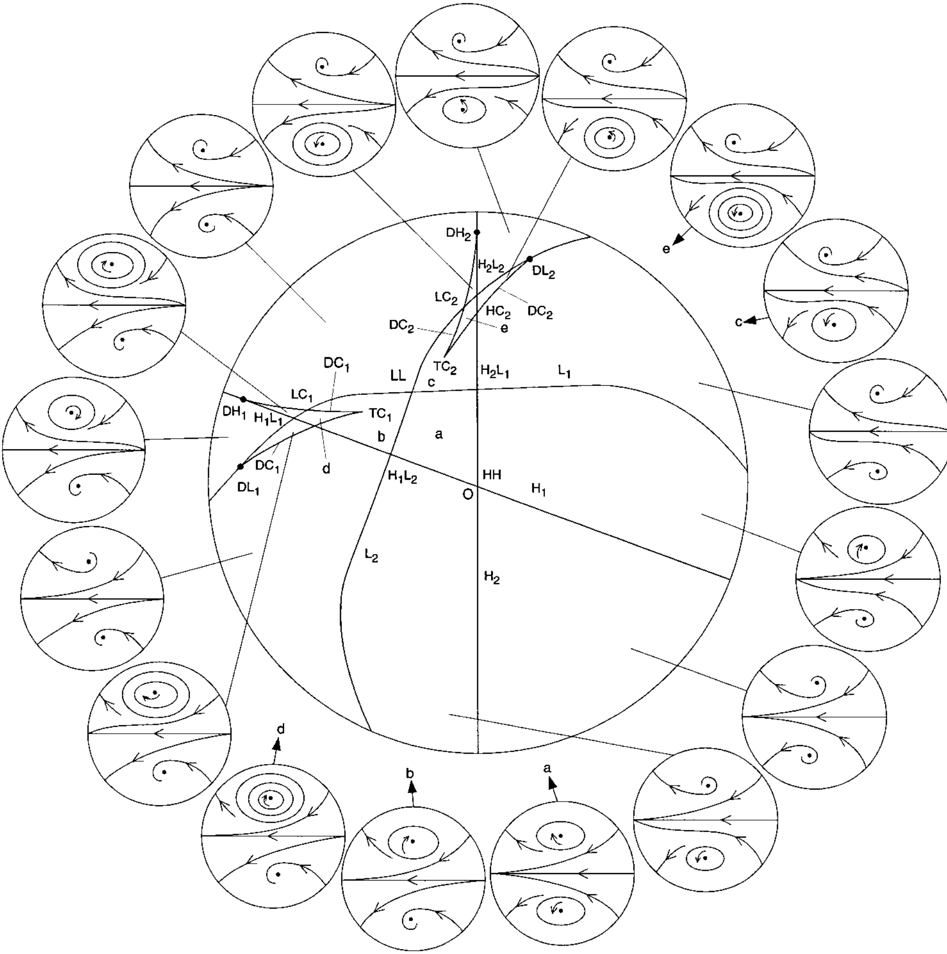

FIG. 2. Bifurcation diagram and structurally stable phase portraits.

$\alpha + \beta P(h) + \gamma Q(h)$, where α, β and γ are parameters depending on μ, and both P and Q are ratios of two Abelian integrals. In Sections 3 and 4 we study the Hopf and heteroclinic bifurcations respectively. Then we prove the monotonicities of P and Q in Section 5. We deduce the Picard–Fuchs equations and the differential equations of P and Q in Section 6. In Section 7 we determine the shape of the curve $\Omega = \{(P, Q)(h) \mid h \in (h_1, h_2)\}$ and prove the uniqueness of its inflection point. We investigate the multiple limit cycle bifurcations in Section 8; and finally, putting the results together, finish the proof of Theorem 1.2 in Section 9. Specific non-trivial technical problems are treated in the appendices.

2. PRELIMINARIES

By the change

$$(\mu_1, \mu_2, \mu_3) = (\delta\nu_1, \delta\nu_2, \delta\nu_3),$$

with $\delta > 0$ small, $(1.5)_\mu$ is transformed to the form

$$\begin{cases} \dot{x} = -y - 3x^2 + y^2 + \delta(\nu_1 x + \nu_2 xy), \\ \dot{y} = x(1-2y) + \delta(\nu_3 x^2). \end{cases} \tag*{$(2.1)_\delta$}$$

Under the scaling,

$$(\nu_1, \nu_2, \nu_3) \to (r\nu_1, r\nu_2, r\nu_3),$$

with $r > 0$, $(2.1)_\delta$ becomes $(2.1)_{r\delta}$. We will show in paragraphs 3, 4 and 8 that the conclusions concerning the bifurcation diagrams are the same for $(2.1)_\delta$ with different δ, if $0 < \delta \ll 1$. Hence the bifurcation diagram of $(2.1)_\delta$ $(\delta > 0)$ has a cone-like structure in ν-space with $\nu \neq 0$. We also observe that by the change of coordinates $(x, t) \mapsto (-x, -t)$ equation (2.1) (or (1.5)) keeps the same form, but all parameters $\{\nu_i, i = 1, 2, 3\}$ (or $\{\mu_i, i = 1, 2, 3\}$) change their signs. These two facts will give the following result

LEMMA 2.1. *The bifurcation diagram of* $(1.5)_\mu$ *has a cone-like structure for* $\mu \in \mathbb{R}^3$, $0 < |\mu| \ll 1$; *and it is symmetric with respect to* $\mu = 0$.

Thus, we will restrict our study of bifurcation diagram and phase portraits of (2.1) to the half sphere $\{\nu_1^2 + \nu_2^2 + \nu_3^2 = 1, \nu_3 \geqslant 0\}$, or to the faces of a half box: $\{\nu_3 = 1, |\nu_1| \leqslant M, |\nu_2| \leqslant M\}$, $\{\nu_1 = \pm M, |\nu_2| \leqslant M, 0 \leqslant \nu_3 \leqslant 1\}$ and $\{\nu_2 = \pm M, |\nu_1| \leqslant M, 0 \leqslant \nu_3 \leqslant 1\}$ with $M > 0$ large (see [D]).

We note that $(2.1)_\delta$ is a perturbation of $(2.1)_0 \in Q_3^R$, whose phase portrait is shown in Fig. 1. To turn the invariant straight line $y = 1/2$ into a coordinate axis, we first make the change of variables $Y = y - 1/2$, then $(2.1)_\delta$ becomes

$$\begin{cases} \dot{x} = -\frac{1}{4} - 3x^2 + Y^2 + \delta(\nu_1 + \frac{1}{2}\nu_2)x + \delta\nu_2 xY, \\ \dot{Y} = -2xY + \delta\nu_3 x^2. \end{cases} \tag{2.2}$$

Then we make the Poincaré transformation (see Chapter 5 of [ZDHD], for example)

$$x = \frac{u}{z}, \qquad Y = \frac{1}{z}, \qquad dt = z\, d\tau, \tag{2.3}$$

system (2.2) is changed to the form

$$\begin{cases} \dfrac{du}{d\tau} = 1 - \dfrac{1}{4} z^2 - u^2 + \delta \left[v_2 u + \left(v_1 + \dfrac{1}{2} v_2 \right) uz - v_3 u^3 \right], \\ \dfrac{dz}{d\tau} = 2zu - \delta v_3 z u^2. \end{cases} \tag{$(2.4)_\delta$}$$

We note that when we transform the phase portraits of $(2.4)_\delta$ back to that of (2.1), we need to reverse the direction of motion for $z<0$ (see (2.3)).

The following two points are important for our further study :

(I) $(2.1)_\delta$ is a family of quadratic systems unfolding $(2.1)_0 \in Q_3^R \backslash Q_3^H$, but $(2.4)_\delta$ is a family of cubic systems unfolding $(2.4)_0$ which is a Hamiltonian system with first integral

$$H(u, z) = z(u^2 + \tfrac{1}{12} z^2 - 1) = h. \tag{2.5}$$

The phase portrait of $(2.4)_0$ is shown in Fig. 3. These are two centers at $C_1(0, 2)$ and $C_2(0-2)$, and two saddle points at $S_1(-1, 0)$ and $S_2(1, 0)$. When $-\frac{4}{3} < h < 0$ (resp. $0 < h < \frac{4}{3}$) the compact connected components of $\Gamma_h = \{(x, y) \mid H(x, y) = h\}$ surround C_1 (resp. C_2). Γ_h shrinks to C_1 (resp. C_2) as $h \downarrow -\frac{4}{3}$ (resp. $h \uparrow \frac{4}{3}$), and Γ_h expands to the heteroclinic loop Γ_{10} (resp. Γ_{20}) as $h \uparrow 0$ (resp. $h \downarrow 0$).

(II) The unbounded heteroclinic loops of $(2.1)_0$ are transformed into the bounded heteroclinic loops Γ_{10} and Γ_{20} of $(2.4)_0$. They share a heteroclinic orbit joining the two saddles S_1 and S_2 and lying on the axis $\{z=0\}$. We note that the axis $\{z=0\}$ comes from the equator at infinity of Fig. 1 (see (2.3)), hence under perturbations the common part of Γ_{01} and Γ_{02} remains unbroken.

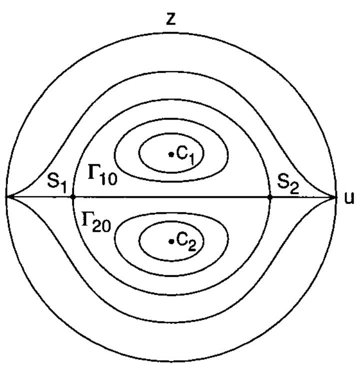

FIG. 3. Phase portrait after Poincaré transformation.

From $(2.4)_\delta$, the Abelian integral $I(h)$ defined in (1.2) is

$$I(h)=\int_{\Gamma_h} (v_2 u+(v_1+\tfrac{1}{2}v_2)\, zu-v_3u^3)\, dz+v_3 zu^2\, du. \tag{2.6}$$

For simplicity of the notations we let

$$I_k(h)=\int_{\Gamma_h} z^k u\, dz, \qquad k=0,\ 1,\ 2,\ 3,\ \ldots \tag{2.7}$$

LEMMA 2.2.

$$I(h)=(v_2-4v_3)\, I_0(h)+(v_1+\tfrac{1}{2}v_2)\, I_1(h)+v_3 I_2(h). \tag{2.8}$$

Proof. Using integration by parts, we have

$$\int_{\Gamma_h} u^3\, dz=-3\int_{\Gamma_h} zu^2\, du. \tag{2.9}$$

Substituting (2.9) into (2.6), we have

$$I(h)=v_2 I_0(h)+(v_1+\tfrac{1}{2}v_2)\, I_1(h)+4v_3\int_{\Gamma_h} zu^2\, du. \tag{2.10}$$

From the equation of $(2.4)_0$, along Γ_h we have

$$(1-\tfrac{1}{4}z^2-u^2)\, dz=2zu\, du, \tag{2.11}$$

multiplying (2.11) by u, then making integration for both sides and using (2.9) again, we get

$$\int_{\Gamma_h} zu^2\, du=-I_0(h)+\tfrac{1}{4}I_2(h). \tag{2.12}$$

Substituting (2.12) into (2.10), we obtain (2.8). ∎

LEMMA 2.3. *Under any perturbations of* $(2.1)_0$ *inside quadratic systems, and transforming it into the* (u, z) *coordinates by the changes* $Y=y-\frac{1}{2}$ *and* (2.3), *the corresponding Abelian integral* $I(h)$ *can be expressed by a linear combination of* $I_0(h)$, $I_1(h)$ *and* $I_2(h)$.

Proof. A straightforward calculation shows that after changing to the (u, z) coordinates, the Abelian integral $I(h)$ must be a linear combination of $I_0(h)$, $I_1(h)$, $I_2(h)$ and the following integrals (along Γ_h):

$$\int z\,dz,\ \int z^2\,dz,\ \int u^2\,dz,\ \int zu^2\,dz,\ \int zu\,du,\ \int z^2u\,du,$$

$$\int u^3\,dz,\ \int z\,du,\ \int zu\,du,\ \int z^2\,du,\ \int z^3\,du.$$

All the integrals in the first line are equal to zero (by using (2.5) and integration by parts), while any integral in the second line can be expressed as a linear combination of $I_0(h)$, $I_1(h)$ and $I_2(h)$ based on the same techniques as used in the proof of Lemma 2.2. ∎

LEMMA 2.4. *To study the bifurcation diagram and phase portraits for all non-conservative perturbations (inside quadratic systems) of* $(1.5)_0$, *it is sufficient to study* $(1.5)_\mu$.

Proof. We need to prove this conclusion from three points of view: Hopf bifurcation (h near $\pm\frac{4}{3}$), heteroclinic bifurcation (h near 0), and the number of zeros of $I(h)$ ($|h| \in [h_1, h_2]$, $0 < h_1 \ll 1$, $0 < \frac{4}{3} - h_2 \ll 1$).

Since both centers of $(1.5)_0$ belong to $Q_3^R \backslash Q_4$, their cyclicity is two (see Theorem 3 of [Z1], for example), and we will prove in Section 3 that $(1.5)_\mu$ gives a full description of the codimension 2 Hopf bifurcation for these two singularities. Hence $(1.5)_\mu$ is versal concerning Hopf bifurcation.

We will prove in Section 4 that the heteroclinic loop bifurcations can appear only along the curves L_i, $i = 1, 2$, with codimension 1 for $L_i \backslash \{DL_i\}$ and codimension 2 for $\{DL_i\}$. The three parameter unfolding $(1.5)_\mu$ will reveal to be versal among all perturbations of $(1.5)_0$ concerning the point of view of heteroclinic bifurcations.

Finally, by Lemma 2.3, for all perturbations of $(1.5)_0$ inside quadratic systems, the Abelian integral $I(h)$ can be expressed as a linear combination of $I_0(h)$, $I_1(h)$ and $I_2(h)$. On the other hand, the integral $I(h)$ related to $(1.5)_\mu$ has the expansion (2.8) (see Lemma 2.2), and the determinant

$$\frac{D[(\nu_2 - 4\nu_3), (\nu_1 + \frac{1}{2}\nu_2), \nu_3]}{D[\nu_1, \nu_2, \nu_3]} \neq 0,$$

hence it is sufficient to study the number of zeros of $I(h)$ ($|h| \in [h_1, h_2]$) only for the perturbed system $(1.5)_\mu$. ∎

Now we consider $(1.5)_\mu$, or equivalently, $(2.4)_\delta$ for (ν_1, ν_2, ν_3) at the faces of the half box as mentioned above. The most interesting phenomena happen on the face $\{v_3=1,\ |v_1|\leqslant M,\ |v_2|\leqslant M\}$ with M large. If we take $v_3=1$, then (2.8) becomes

$$I(h)=(v_2-4)\,I_0(h)+(v_1+\tfrac{1}{2}v_2)\,I_1(h)+I_2(h). \tag{2.13}$$

Since $I_0(h)=\int_{\Gamma_h} u\,dz=\iint_{H\leqslant h} dz\,du>0$ for $h\neq\pm\frac{4}{3}$, the number of zeros of (2.13) is the same as for the function

$$M(h)=(v_2-4)+(v_1+\tfrac{1}{2}v_2)\,P(h)+Q(h), \tag{2.14}$$

where

$$P(h)=\frac{I_1(h)}{I_0(h)},\qquad Q(h)=\frac{I_2(h)}{I_0(h)},\qquad |h|\in(0,\ \tfrac{4}{3}). \tag{2.15}$$

The region of definition of P and Q can be extended to $-\frac{4}{3}\leqslant h\leqslant\frac{4}{3}$, if we define the values of the functions P and Q at $\pm\frac{3}{4}\mp 0$ and 0 ± 0 by corresponding limits as $h\to\pm\frac{4}{3}\mp 0$, or $h\to 0\pm 0$. In fact, by direct calculations we have the following result.

LEMMA 2.5.

$$\lim_{h\to 0+0} P(h)=-\frac{8}{\sqrt{3}\pi},\qquad \lim_{h\to(4/3)-0} P(h)=-2;$$

$$\lim_{h\to 0-0} P(h)=\frac{8}{\sqrt{3}\pi},\qquad \lim_{h\to-(4/3)+0} P(h)=2;$$

$$\lim_{h\to 0\pm 0} Q(h)=3,\qquad \lim_{h\to\pm(4/3)\mp 0} Q(h)=4.$$

Besides, it is easy to see that

$$P\in C^{\circ}([-\tfrac{4}{3},\ 0)\cup(0,\ \tfrac{4}{3}])\cap C^{\infty}((-\tfrac{4}{3},\ 0)\cup(0,\ \tfrac{4}{3})),$$
$$Q\in C^{\circ}[-\tfrac{4}{3},\ \tfrac{4}{3}]\cap C^{\infty}((-\tfrac{4}{3},\ 0)\cup(0,\ \tfrac{4}{3})).$$

If we prove that $P=P(h)$ is monotonic for $h\in(0,\ \pm\frac{4}{3})$, then we can take P as new parameter and, instead of (2.14), consider the number of zeros of the function

$$(v_2-4)+(v_1+\tfrac{1}{2}v_2)P+\tilde{Q}(P), \tag{2.16}$$

where $\tilde{Q}(P)=Q(h(P))$, $h=h(P)$ is the inverse function of $P=P(h)$. This is equivalent to consider a geometric problem in PQ-plane: finding the number of intersection points of the straight lines $Q = -(v_1 + \frac{1}{2}v_2)P - (v_2 - 4)$ and the curve $Q = \tilde{Q}(P)$ for different v_1 and v_2. This is a key point of the paper.

3. HOPF BIFURCATIONS

In this section it is convenient to consider $(2.1)_\delta$ directly. We note that the singularity $(0, 0)$ (resp. $(0, 1)$) of $(2.1)_\delta$ corresponds to the singularity $C_2(0, -2)$ (resp. $C_1(0, 2)$) of $(2.4)_\delta$.

We first consider the point $(0, 0)$ of (2.1). It is obvious that the necessary condition for Hopf bifurcation is $v_1 = 0$. Suppose $v_1 = 0$, then by the formulas in [L], the first two Lyapunov constants are

$$\begin{aligned} W_1 &= 2\delta(4v_3 - v_2), \\ W_2 &= -\delta v_2 v_3 (5v_3 - v_2)(4 - 7\delta^2 v_3^2). \end{aligned} \tag{3.1}$$

If $0 < \delta \ll 1$, $v_3 > 0$, then $W_1 = 0$ implies $W_2 < 0$.

Next, we consider the Hopf bifurcation at point $(0, 1)$ of (2.1). By the change of coordinates $\bar{x} = x$, $\bar{y} = -(y - 1)$, $(2.1)_\delta$ becomes

$$\begin{cases} \dot{\bar{x}} = -\bar{y} + \delta(v_1 + v_2)\bar{x} - 3\bar{x}^2 - \delta v_2 \bar{x}\bar{y} + \bar{y}^2, \\ \dot{\bar{y}} = \bar{x} - \delta v_3 \bar{x}^2 - 2\bar{x}\bar{y}. \end{cases}$$

Hence, the necessary condition for Hopf bifurcation is $v_1 + v_2 = 0$. Under this condition, the first two Lyapunov constants are

$$\overline{W_1} = -W_1; \qquad \overline{W_2} = -W_2.$$

Therefore, we obtain the following result.

THEOREM 3.1. *Suppose that* $0 < \delta \ll 1$, $v_3 > 0$. *Then system* $(2.1)_\delta$ *at singularity* $(0, 0)$ (*resp.* $(0, 1)$) *has a Hopf bifurcation of order* 1 *if* $v_1 = 0$, $v_2 \neq 4v_3$ (*resp.* $v_1 + v_2 = 0$, $v_2 \neq 4v_3$), *of order* 2 *if* $v_1 = v_2 - 4v_3 = 0$ (*resp.* $v_1 + v_2 = v_2 - 4v_3 = 0$). *Two is the highest order of Hopf bifurcation. The singular point* $(0, 0)$ (*resp.* $(0, 1)$) *is stable if* $v_1 < 0$, *or* $v_1 = 0$ *and* $v_2 \geqslant 4v_3$ (*resp.* $v_1 + v_2 > 0$, *or* $v_1 + v_2 = 0$ *and* $v_2 \geqslant 4v_3$), *is unstable if* $v_1 > 0$, *or* $v_1 = 0$ *and* $v_2 < 4v_3$ (*resp.* $v_1 + v_2 < 0$, *or* $v_1 + v_2 = 0$ *and* $v_2 < 4v_3$).

We remark here that by using Theorem 3.1, we get the codimension 1 and 2 Hopf bifurcation diagram in Fig. 5, for $\delta \to 0$, on domains shrinking to zero. Concerning a uniform knowledge for $0 < \delta \ll 1$, we give the necessary explanation at the end of Section 9.

4. HETEROCLINIC LOOP BIFURCATIONS

LEMMA 4.1. *The necessary condition for existence of heteroclinic loops of* $(2.1)_\delta$, *surrounding the singular points* $(0, 1)$ *and* $(0, 0)$, *are respectively*

$$L_1 : 8v_1 + (\sqrt{3}\pi + 4)\, v_2 - \sqrt{3}\pi v_3 + O(\delta) = 0, \tag{4.1}$$

and

$$L_2 : 8v_1 - (\sqrt{3}\pi - 4)\, v_2 + \sqrt{3}\pi v_3 + O(\delta) = 0. \tag{4.2}$$

Proof. As seen in Section 2, the study of the unbounded heteroclinic loop bifurcations of $(2.1)_\delta$, is equivalent to the study of the bounded heteroclinic loop bifurcations of $(2.4)_\delta$ with the property that under perturbations the common part of the two loops is not broken.

Hence, the necessary condition is

$$\lim_{h \to 0 \mp 0} I(h) + O(\delta) = 0, \tag{4.3}$$

where $I(h)$ is given in (2.8), and

$$\lim_{h \to 0-0} I_0(h) = 2 \int_0^{2\sqrt{3}} \left(1 - \frac{z^2}{12}\right)^{1/2} dz = \sqrt{3}\pi,$$

$$\lim_{h \to 0-0} I_1(h) = 2 \int_0^{2\sqrt{3}} z\left(1 - \frac{z^2}{12}\right)^{1/2} dz = 8,$$

$$\lim_{h \to 0-0} I_2(h) = 2 \int_0^{2\sqrt{3}} z^2\left(1 - \frac{z^2}{12}\right)^{1/2} dz = 3\sqrt{3}\pi,$$

$$\lim_{h \to 0+0} I_k(h) = (-1)^{k+1} \lim_{h \to 0-0} I_k(h), \qquad k = 0, 1, 2.$$

Entering the above values into (4.3), we obtain (4.1) and (4.2). ∎

For systems $(2.4)_\delta$, we denote by r_1 and r_2 the ratios of hyperbolicity of the two respective saddle points. Recall that the ratio of hyperbolicity of a saddle point is defined by

$$r = \left|\frac{\lambda_1}{\lambda_2}\right|, \tag{4.4}$$

where $\lambda_1 < 0$ and $\lambda_2 > 0$ are the eigenvalues of the linear part of the vector field at this point.

LEMMA 4.2 *Suppose that* $0<\delta\ll 1$, $v_3>0$, *then*

$$r_1 r_2 = 1 + (4v_3 - v_2)\delta + \frac{(4v_3 - v_2)^2}{2}\delta^2 + O(\delta^3). \tag{4.5}$$

Proof. The saddle points of $(2.4)_\delta$ have coordinates $(u_i, 0)$, $i=1, 2$, satisfying

$$\delta v_3 u_i^3 + u_i^2 - \delta v_2 u_i - 1 = 0, \qquad i=1, 2. \tag{4.6}$$

Hence

$$\begin{cases} u_1 = -1 + \dfrac{v_2 - v_3}{2}\delta - \dfrac{(v_2 - v_3)(v_2 - 5v_3)}{8}\delta^2 + O(\delta^3), \\ u_2 = 1 + \dfrac{v_2 - v_3}{2}\delta + \dfrac{(v_2 - v_3)(v_2 - 5v_3)}{8}\delta^2 + O(\delta^3). \end{cases} \tag{4.7}$$

On the other hand, at $(u_i, 0)$ the linear part of $(2.4)_\delta$ is given by the matrix

$$\begin{pmatrix} -2u_i + \delta v_2 - 3\delta v_3 u_i^2 & * \\ 0 & 2u_i - \delta v_3 u_i^2 \end{pmatrix}.$$

Therefore,

$$r_1 r_2 = \frac{-2u_1 + \delta v_3 u_1^2}{-2u_1 + \delta v_2 - 3\delta v_3 u_1^2}\;\frac{2u_2 - \delta v_2 + 3\delta v_3 u_2^2}{2u_2 - \delta v_3 u_2^2}.$$

Substituting (4.7) into the above equality, we obtain (4.5). ∎

THEOREM 4.3. *If* $0<\delta\ll 1$, $v_3\neq 0$, *then system* $(2.1)_\delta$ *has a heteroclinic bifurcation of codimension* 1 *along* $L_i\backslash\{DL_i\}$, *of codimension* 2 *at* $\{DL_i\}$, $i=1, 2$, *and two is the highest codimension. The corresponding bifurcation diagram is shown in Figure* 2.

Proof. By Lemma 4.1, for $0<\delta\ll 1$, along L_i system $(2.4)_\delta$ has a heteroclinic loop Γ_{i0}. We consider the case Γ_{20} (the case Γ_{10} is similar). Let σ_i, τ_i $(1\leqslant i\leqslant 2)$ be segments transverse to the vector field $(2.4)_\delta$ near the saddle points S_1 and S_2 (see Fig. 4), and we parametrize σ_1 by h given by (2.5), hence $h\geqslant 0$.

The flow of $(2.4)_\delta$ induces Dulac maps D_1 and D_2, and regular transitions R_1 and R_2. Since one of the saddle connections, namely a part of u-axis $\{(z, u)\,|\,z=0,\ -1\leqslant u\leqslant 1\}$, is fixed for any δ, the composition of the two

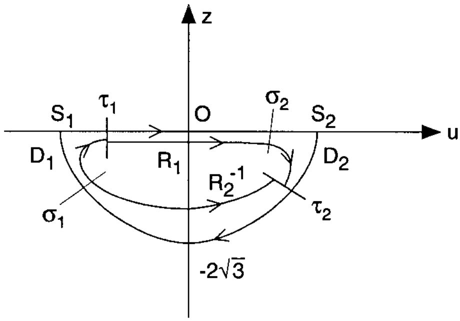

FIG. 4. Transition mappings.

Dulac maps and the regular map R_1 can be written in the form (see for example Lemma 5.1 of [DRR])

$$D_2 \circ R_1 \circ D_1(h) = A(\lambda)\, h^{r_1(\lambda)\, r_2(\lambda)}(1 + \varphi(h, \lambda)), \tag{4.8}$$

where $\lambda = (\delta, v_i)$, $A(\lambda)$ is C^∞ and $A|_{\delta \to 0} \neq 0$, $\varphi(h, \lambda)$ is C^∞ for $(h, \lambda) \in (0, \varepsilon] \times W$, where $\varepsilon > 0$, W is some neighborhood of the original value of λ in λ-space, and φ satisfies the property

$$\forall n \in \mathbb{Z}, \qquad \lim_{h \to 0} h^n \frac{\partial^n \varphi}{\partial h^n}(h, \lambda) = 0 \text{ uniformly in } \lambda \in W.$$

For more details, see [DER].

Hence the displacement on τ_2 is given by

$$\Psi(h, \lambda) = D_2 \circ R_1 \circ D_1(h) - R_2^{-1}(h), \tag{4.9}$$

where

$$R_2^{-1}(h) = a_0(\lambda) + a_1(\lambda) h + \cdots. \tag{4.10}$$

Let

$$r_1(\lambda)\, r_2(\lambda) = 1 - \alpha(\lambda), \tag{4.11}$$

and the compensator

$$\omega(h, \lambda) = \begin{cases} \dfrac{h^{-\alpha(\lambda)} - 1}{\alpha(\lambda)} & \text{if } \alpha(\lambda) \neq 0, \\ -\ln h & \text{if } \alpha(\lambda) = 0. \end{cases}$$

Then

$$h^{r_1(\lambda)\, r_2(\lambda)} = h^{1-\alpha(\lambda)} = (1+\alpha(\lambda)\,\omega)\, h. \tag{4.12}$$

Subsituting (4.8), (4.12) and (4.10) into (4.9), we have

$$\Psi(h, \lambda) = c_0(\lambda) + c_1(\lambda)\, h\omega(1+\varphi(h, \lambda)) + c_2(\lambda)\, h(1+\psi(h, \lambda)) + \cdots, \tag{4.13}$$

where $c_1(\lambda) = A(\lambda)\alpha(\lambda)$, and $\psi(h, \lambda)$ has the same property as $\varphi(h, \lambda)$. By (4.11) and (4.5),

$$c_1(\lambda) = A(\lambda)((v_2 - 4v_3) - \frac{(v_2 - 4v_3)^2}{2}\,\delta + O(\delta^2))\,\delta. \tag{4.14}$$

On the other hand, we know that

$$\Psi(h, \lambda) = \delta I(h) + O(\delta^2).$$

Hence

$$\begin{cases} c_0(\lambda) = \delta I(0+) + O(\delta^2), \\ c_2(\lambda) = \delta I'(0+) + O(\delta^2), \qquad \text{if} \quad c_1(\lambda) \equiv 0. \end{cases} \tag{4.15}$$

From (4.14) we know that for $0 < \delta \ll 1$, there exists a function

$$v_2 = v_2(v_3, \delta) = 4v_3 + O(\delta^2), \tag{4.16}$$

such that $c_1(\lambda)|_{v_2 = v_2(v_3,\ \delta)} \equiv 0$. For any fixed $v_3 \neq 0$, (4.16) determines a curve in $v_1 v_2$-plane, which intersects L_i at $\{DL_i\}$ transversally.

Thus, along $L_i \backslash \{DL_i\}$ we have $c_0(\lambda) = 0$ and $c_1(\lambda) \neq 0$. We will prove that at $\{DL_i\}$, $c_0(\lambda) = c_1(\lambda) = 0$ and $c_2(\lambda) \neq 0$. Therefore, by using a derivation-division algorithm (see [R] and [DER]), we conclude that near $L_i \backslash \{DL_i\}$ (resp. $\{DL_i\}$), $\Psi(h, \lambda)$ has at most one (resp. two) zero(s) for h near 0. This implies the desired results.

Now suppose that $c_0(\lambda) = c_1(\lambda) = 0$ i.e. $I(0+) = 0$, $v_2 = 4v_3 + O(\delta^2)$. By (2.8)

$$I(h) = (v_1 + 2v_3)\, I_1(h) + v_3 I_2(h) + O(\delta^2), \tag{4.17}$$

where

$$I_k(h) = \int_{\Gamma_h} z^k u\, dz, \qquad k = 1, 2. \tag{4.18}$$

From (2.5) we know that along Γ_h, $\partial u/\partial h = 1/2zu$. Hence $I'_k(h) = \int_{\Gamma_h} z^k/2zu\, dz$, $k = 1, 2$, and

$$I'(h) = (v_1 + 2v_3)\, I'_1(h) + v_3 I'_2(h) + O(\delta^2), \tag{4.19}$$

where

$$\begin{cases} I'_1(0+) = \displaystyle\int_0^{-2\sqrt{3}} \left(1 - \frac{z^2}{12}\right)^{-1/2} dz = -\sqrt{3}\pi, \\ I'_2(0+) = \displaystyle\int_0^{-2\sqrt{3}} z\left(1 - \frac{z^2}{12}\right)^{1/2} dz = 12. \end{cases} \tag{4.20}$$

By (4.2), (4.19) and (4.20) we have

$$I(0+) = 8v_1 + (16 - 3\sqrt{3}\pi)\, v_3 + O(\delta^2),$$
$$I'(0+) = -\sqrt{3}\pi v_1 + (12 - 2\sqrt{3}\pi)\, v_3 + O(\delta^2).$$

It is obvious that for $0 < \delta \ll 1$, $I(0+) = 0$ implies $I'(0+) \neq 0$, and by (4.15), $c_0(\lambda) = c_1(\lambda) = 0$ implies $c_2(\lambda) \neq 0$. ∎

Summing up the results in Sections 3 and 4, we can draw the bifurcation diagram of Hopf and heteroclinic loop bifurcations on the face $\{v_3 = 1, |v_1| \leqslant M, |v_2| \leqslant M\}$ with M large, shown in Fig. 5.

It is easy to find from these two sections that on the faces $\{v_1 = \pm M, |v_2| \leqslant M, 0 \leqslant v_3 \leqslant 1\}$ and $\{v_2 = \pm M, |v_1| \leqslant M, 0 \leqslant v_3 \leqslant 1\}$ with M large

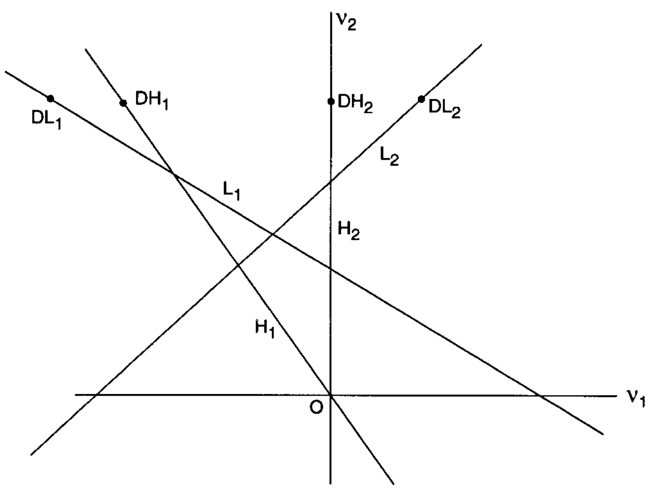

FIG. 5. Limiting bifurcation diagram.

there are only Hopf and heteroclinic bifurcations of order 1, and no such bifurcations of order 2 or higher.

5. MONOTONICITIES OF $P(h)$ AND $Q(h)$

From (2.5) we know that for each $h \in (-\frac{4}{3}, 0) \cup (0, \frac{4}{3})$, Γ_h is symmetric with respect to the axis $u=0$, and Γ_h and Γ_{-h} are symmetric with respect the axis $z=0$. Hence, by (2.7), we have

$$I_k(h) = (-1)^{k+1} I_k(-h);$$

by the definition (2.15), we know that

$$P(-h) = -P(h), \qquad Q(-h) = Q(h). \tag{5.1}$$

Thus, we will only consider the case $h \in (-\frac{4}{3}, 0)$. In this case, the curves $\{\Gamma_h \mid -\frac{4}{3} < h < 0\}$ surround the center (0, 2), we always have $z > 0$.

Write the first integral (2.5) in the form

$$zu^2 - \Phi(z) = h, \tag{5.2}$$

where $z > 0$, $\Phi(z) = z - \frac{1}{12}z^3$, satisfying

$$\Phi'(z)(z-2) < 0 \qquad \text{for} \quad z \neq 2. \tag{5.3}$$

For any $z \in (0, 2)$, there is an unique $\tilde{z} \in (2, \sqrt{12})$ such that

$$\Phi(z) = \Phi(\tilde{z}), \qquad 0 < z < 2 < \tilde{z} < \sqrt{12}. \tag{5.4}$$

Therefore, we can define a function $\tilde{z} = \tilde{z}(z)$ for $0 < z < 2$ satisfying (5.4). By (5.3) we have

$$\frac{d\tilde{z}}{dz} = \frac{\Phi'(z)}{\Phi'(\tilde{z})} < 0. \tag{5.5}$$

Let

$$\alpha = z + \tilde{z}, \qquad \beta = z\tilde{z}, \tag{5.6}$$

where $\tilde{z} = \tilde{z}(z)$ is defined as above, $0 < z < 2$. It is obvious that $\alpha > 0$ and $\beta > 0$. We can obtain a precise estimate of α.

LEMMA 5.1. *For* $0<z<2$ *we have that*

(i) $0<\Phi'(z)< -\Phi'(\tilde{z})$.

(ii)
$$\beta=\alpha^2-12 \qquad \textit{and} \qquad \sqrt{12}<\alpha<4. \tag{5.7}$$

Proof. From (5.4) we get

$$(\tilde{z}-z)[1-\tfrac{1}{12}(\tilde{z}^2+z\tilde{z}+z^2)]=0. \tag{5.8}$$

Since $\tilde{z}-z>0$ and $\tilde{z}^2+\tilde{z}z+z^2=\alpha^2-\beta$, we obtain $\beta=\alpha^2-12$ from (5.8). Note that $z\to 0$ implies $\tilde{z}\to\sqrt{12}$ and $\alpha\to\sqrt{12}$. Consider $d\alpha/dz=1+d\tilde{z}/dz=1/\Phi'(\tilde{z})\ (\Phi'(\tilde{z})+\Phi'(z))=1/\Phi'(\tilde{z})\ [2-\frac{1}{4}(z^2+\tilde{z}^2)]$.

Using $z^2+\tilde{z}^2=\alpha^2-2\beta$ and $\beta=\alpha^2-12$ we obtain

$$\frac{d\alpha}{dz}=\frac{1}{4\Phi'(\tilde{z})}\,(\alpha^2-16).$$

Since $\Phi'(\tilde{z})<0$ (see (5.3)), we have $d\alpha/dz>0$ for $\alpha<4$. At $\alpha=4$ we have $d\alpha/dz=0$ and hence $z^2+\tilde{z}^2=8$; as such $z=2$. Hence $d\alpha/dz>0$ for $0<z<2$, and we must have $\sqrt{12}<\alpha<4$, which also implies $\Phi'(\tilde{z})+\Phi'(z)<0$. ∎

Let

$$F(z)=\tilde{z}(\tilde{z}^2-4)^4-z(z^2-4)^4,$$

$$G(z)=\tilde{z}(\tilde{z}^2-4)^2\,(4-5z^2)-z(z^2-4)^2\,(4-5\tilde{z}^2),$$

$$H(z)=4z\tilde{z}(\tilde{z}(4-\tilde{z}^2)(4-z^2)^2+z(4-z^2)(4-\tilde{z}^2)^2)+(\tilde{z}^2-z^2)\,G(z),$$

where $\tilde{z}=\tilde{z}(z)$, $0<z<2$.

LEMMA 5.2. *If* $F(z)>0$, $G(z)>0$ *and* $H(z)>0$ *for* $0<z<2$, *then* $P'(h)<0$ *and* $Q'(h)<0$ *for* $-\frac{4}{3}<h<0$.

Proof. Define

$$\zeta_k(z)=\frac{z^k\Psi(\tilde{z})-\tilde{z}^k\Psi(z)}{\Psi(\tilde{z})-\Psi(z)},$$

where $\Psi(z)=z^{1/2}\Phi'(z)$, $\tilde{z}=\tilde{z}(z)$, $0<z<2$, and $k=1,\ 2$. By Theorem 2 of [LZ] if $\zeta_k'(z)>0$ for $0<z<2$ and $k=1,\ 2$, then the desired result follows. Calculation shows that

$$\zeta_k'(z)=\frac{\tilde{z}^{1/2}}{(\Psi(\tilde{z})-\Psi(z))^2}\,\frac{W_k(z)}{\Psi(\tilde{z})}, \qquad k=1,\ 2,$$

where

$$W_k(z) = k(\Psi(\tilde{z}) - \Psi(z))[z^{k-1}\tilde{z}^{-1/2}(\Psi(\tilde{z}))^2 - \tilde{z}^{k-1}z^{-1/2}(\psi(z))^2]$$
$$+ (z^k - \tilde{z}^k)[\tilde{z}^{-1/2}(\psi(\tilde{z}))^2\, \Psi'(z) - z^{-1/2}(\Psi(z))^2\, \Psi'(\tilde{z})].$$

Since $0 < z < 2 < \tilde{z}$, $\Psi(\tilde{z}) < 0 < \Psi(z)$, to prove $\zeta_1'(z) > 0$ it is sufficient to show that both of the last factors of the two terms in $W_1(z)$ are positive. The first one is positive if $F(z) > 0$ (taking $\Psi(z) = z^{1/2}(1 - \frac{1}{4}z^2)$); the second one equals to $(z\tilde{z})^{-1/2}G(z)/128$. To prove $\zeta_2'(z) > 0$, we rewrite $W_2(z)$ as follows:

$$\begin{aligned} W_2(z) &= 2z\tilde{z}[(\Phi'(\tilde{z}))^3 + (\Phi'(z))^3] - 2(\tilde{z}z)^{1/2}\,[\tilde{z}\Phi'(\tilde{z})(\Phi'(z))^2 \\ &\quad + z\Phi'(z)(\Phi'(\tilde{z}))^2] \\ &\quad + (z^2 - \tilde{z}^2)[\tilde{z}^{-1/2}(\Psi(\tilde{z}))^2\, \Psi'(z) - z^{-1/2}(\Psi(z))^2\, \Psi'(\tilde{z})] \\ &= 2z\tilde{z}[(\Phi'(\tilde{z}))^3 + (\Phi'(z))^3] - \tfrac{1}{128}(z\tilde{z})^{-1/2}\, H(z). \end{aligned}$$

By Lemma 5.1(i), the first term is negative, hence $W_2(z) < 0$ if $H(z) > 0$. This finishes the proof of the lemma. ∎

THEOREM 5.3 *For* $-\frac{4}{3} < h < 0$, *we have that* $P'(h) < 0$ *and* $Q'(h) < 0$.

Proof. By Lemma 5.2, we only need to prove that $F(z) > 0$, $G(z) > 0$ and $H(z) > 0$ for $0 < z < 2$.

Let $A_k = \sum_{i=0}^{k} \tilde{z}^{k-i}z^k$, then it is easy to verify that

$$\begin{cases} A_2 = \alpha^2 - \beta, \\ A_4 = \alpha^4 - 3\alpha^2\beta + \beta^2, \\ A_6 = \alpha^6 - 5\alpha^4\beta + 6\alpha^2\beta^2 - \beta^3, \\ A_8 = \alpha^8 - 7\alpha^6\beta + 15\alpha^4\beta^2 - 10\alpha^2\beta^3 + \beta^4, \end{cases} \tag{5.9}$$

where α and β are defined in (5.6). Hence

$$\begin{aligned} F(z) &= \tilde{z}(\tilde{z}^2 - 4)^4 - z(z^2 - 4)^4 \\ &= (\tilde{z} - z)(A_8 - 16A_6 + 96A_4 - 256A_2 + 256). \end{aligned}$$

Substituting (5.9) into the above expression and taking $\beta = \alpha^2 - 12$ (see Lemma 5.1(ii)), we obtain

$$F(z) = 4(\tilde{z} - z)(\alpha^2 - 16)^2(5\alpha^2 + 4) > 0,$$

since $12 < \alpha^2 < 16$ for $0 < z < 2$ (Lemma 5.1(ii)).

Similarly, we have

$$\begin{aligned} G(z) &= (-5\tilde{z}^5 z^2 + 4\tilde{z}^5 + 40\tilde{z}^3 z^2 - 32\tilde{z}^3 - 80\tilde{z}z^2 + 64\tilde{z}) \\ &\quad - (-5z^5\tilde{z}^2 + 4z^5 + 40z^3\tilde{z}^2 - 32z^3 - 80z\tilde{z}^2 + 64z) \\ &= (\tilde{z} - z)[-5\beta^2(\alpha^2 - \beta) + 4(\alpha^4 - 3\alpha^2\beta + \beta^2) \\ &\quad + 40\beta^2 - 32(\alpha^2 - \beta) + 80\beta + 64]. \end{aligned}$$

taking $\beta = \alpha^2 - 12$ we have

$$G(z) = 8(\tilde{z} - z)(16 - \alpha^2)(3\alpha^2 - 28) > 0. \tag{5.10}$$

To compute $H(z)$, we first have

$$\begin{aligned} &\tilde{z}(4 - \tilde{z}^2)(4 - z^2)^2 + z(4 - z^2)(4 - \tilde{z}^2)^2 \\ &= -\alpha\beta^3 + 8\alpha\beta^2 - 16\alpha(\alpha^2 - 3\beta) + 4\alpha\beta(\alpha^2 - 3\beta) - 32\alpha\beta + 64\alpha \\ &= -\alpha(\alpha^2 - 4)(\alpha^2 - 16)^2. \end{aligned}$$

Note that

$$(\tilde{z}^2 - z^2)(\tilde{z} - z) = (\tilde{z} + z)(\tilde{z} - z)^2 = \alpha(\alpha^2 - 4\beta) = 3\alpha(16 - \alpha^2).$$

Hence, by using (5.10) we finally obtain

$$\begin{aligned} H(z) &= 4(\alpha^2 - 12)[-\alpha(\alpha^2 - 4)(\alpha^2 - 16)^2] + 24\alpha(\alpha^2 - 16)^2\,(3\alpha^2 - 28) \\ &= 4\alpha(\alpha^2 - 16)^2\,(-\alpha^4 + 34\alpha^2 - 216) > 0 \end{aligned}$$

for $\sqrt{12} < \alpha < 4$. ∎

6. PICARD–FUCHS EQUATIONS AND RELEVANT RESULTS

LEMMA 6.1. *$I_0(h)$, $I_1(h)$ and $I_2(h)$, defined in* (2.7), *satisfy the following Picard–Fuchs equations*

$$\frac{d}{dh}\begin{pmatrix} I_0 \\ I_1 \\ I_2 \end{pmatrix} = \frac{1}{D(h)} \begin{pmatrix} \frac{3}{4}h^2 - 2 & -\frac{3}{4}h & \frac{2}{3} \\ h & \frac{9}{8}h^2 & -h \\ -\frac{3}{2}h^2 & -3h & \frac{3}{2}h^2 \end{pmatrix} \begin{pmatrix} I_0 \\ I_1 \\ I_2 \end{pmatrix}, \tag{6.1}$$

where

$$D(h) = \tfrac{9}{8}h[h^2 - (\tfrac{4}{3})^2]. \tag{6.2}$$

Proof. From (2.5) the function $u = u(z, h)$ satisfies

$$\frac{\partial u}{\partial h} = \frac{1}{2zu}. \tag{6.3}$$

Hence

$$I'_k(h) = \int_{\Gamma_h} \frac{z^k}{2zu}\, dz, \qquad k = 0, 1, 2, 3... . \tag{6.4}$$

Using (6.4) and (2.5) again, we have

$$\begin{aligned} I_k &= \int_{\Gamma_h} z^k u\, dz = \int_{\Gamma_h} \frac{2z^{k+1}u^2}{2zu}\, dz = \int_{\Gamma_h} \frac{2z^k(h + z - (1/12)z^3)}{2zu}\, dz \\ &= 2hI'_k + 2I'_{k+1} - \tfrac{1}{6} I'_{k+3}. \end{aligned} \tag{6.5}$$

On the other hand, using integration by parts and the equation of $(2.4)_0$, we have

$$\begin{aligned} I_k &= \int_{\Gamma_h} z^k u\, dz = -\frac{1}{k+1} \int_{\Gamma_h} z^{k+1}\, du \\ &= -\frac{1}{k+1} \int_{\Gamma_h} \frac{z^{k+1}(1 - u^2 - \frac{1}{4}z^2)}{2zu}\, dz \\ &= -\frac{1}{k+1} I'_{k+1} + \frac{1}{2(k+1)} I_k + \frac{1}{4(k+1)} I'_{k+3}. \end{aligned} \tag{6.6}$$

Removing I_{k+3} from (6.5) and (6.6), we obtain

$$(k+2) I_k = 3hI'_k + 2I'_{k+1}. \tag{6.7}$$

Taking $k = 0, 1, 2$ respectively, we have

$$\begin{cases} I_0 = \frac{3}{2}hI'_0 + I'_1, \\ I_1 = hI'_1 + \frac{2}{3}I'_2, \\ I_2 = \frac{3}{4}hI'_2 + \frac{1}{2}I'_3. \end{cases} \tag{6.8}$$

Taking $k = 0$ in (6.5) and using the first equation of (6.8), we have

$$I'_3 = 3hI'_0 + 6I'_1.$$

Substituing the above equality into the third equation of (6.8), we obtain

$$\begin{pmatrix} I_0 \\ I_1 \\ I_2 \end{pmatrix} = \begin{pmatrix} \frac{3}{2}h & 1 & 0 \\ 0 & h & \frac{2}{3} \\ \frac{3}{2}h & 3 & \frac{3}{4}h \end{pmatrix} \begin{pmatrix} I'_0 \\ I'_1 \\ I'_2 \end{pmatrix},$$

which implies (6.1). ▮

LEMMA 6.2. *The functions* $P(h)$ *and* $Q(h)$, *defined in* (2.15), *satisfy the following differential equations*

$$\begin{cases} D(h)P' = h + (\frac{3}{8}h^2 + 2)P - hQ + \frac{3}{4}hP^2 - \frac{2}{3}PQ, \\ D(h)Q' = -\frac{3}{2}h^2 - 3hP + (\frac{3}{4}h^2 + 2)Q + \frac{3}{4}hPQ - \frac{2}{3}Q^2, \end{cases} \tag{6.9}$$

where $h \in (-\frac{4}{3}, 0) \cup (0, \frac{4}{3})$, *and* $D(h) = \frac{9}{8}h(h^2 - (\frac{4}{3})^2)$.

Proof. Since

$$P' = \frac{I'_1 I_0 - I'_0 I_1}{I_0^2}, \qquad Q' = \frac{I'_2 I_0 - I'_0 I_2}{I_0^2},$$

Substituting the right hand side of (6.1) into the equations above, we obtain (6.9). ▮

LEMMA 6.3. (i) $P'(h) \to -\infty$, $Q'(h) \to \pm\infty$ *and* $Q'(h)/P'(h) \to \mp(3\sqrt{3}/8)\pi \approx \mp 2.04$ *as* $h \to 0 \pm 0$;

(ii) $P'(h) \to -\frac{1}{4}$, $Q'(h) \to \pm\frac{1}{2}$ *and* $Q'(h)/P'(h) \to \mp 2$ *as* $h \to \pm\frac{4}{3} \mp 0$.

Proof. Considering h as a third variable, we change (6.9) into the 3-dimensional system

$$\begin{cases} \dot{h} = \frac{9}{8}h^3 - 2h = D(h). \\ \dot{P} = h + (\frac{3}{8}h^2 + 2)P - hQ + \frac{3}{4}hP^2 - \frac{2}{3}PQ, \\ \dot{Q} = -\frac{3}{2}h^2 - 3hP + (\frac{3}{4}h^2 + 2)Q + \frac{3}{4}hPQ - \frac{2}{3}Q^2. \end{cases} \tag{6.10}$$

(i) At $(h, P, Q) = (0, 8/\sqrt{3}\pi, 3)$ the linear part of this vector field is given by

$$\begin{pmatrix} -2 & 0 & 0 \\ -2 + \dfrac{16}{\pi^2} & 0 & -\dfrac{16}{3\sqrt{3}\pi} \\ -\dfrac{2\sqrt{3}}{\pi} & 0 & -2 \end{pmatrix}, \tag{6.11}$$

having 0 as a simple eigenvalue and -2 as a double eigenvalue with a 1-dimensional eigenspace. The eigenspace of respectively 0 and -2 are contained in $\{h=0\}$, implying that both $\lim_{h\to 0-0} P'(h)$ and $\lim_{h\to 0-0} Q'(h)$ exist with absolute value $+\infty$.

A similar calculation at $(h, P, Q)=(0, -(8/\sqrt{3\pi}), 3)$ will provide the same conclusion for $h\to 0+0$.

For the rest of the statements, implying that in both situations $(h, P(h), Q(h))$ belongs to the stable manifold at $(0, \pm(8/\sqrt{3\pi}), 3)$, we proceed as follows: from (6.9) we have

$$(\tfrac{9}{8}h^2-2)(QP'-PQ')=Q-Q^2+\tfrac{3}{2}hP+3P^2-\tfrac{3}{8}hPQ.$$

This implies that

$$\lim_{h\to 0}\frac{QP'-PQ'}{PP'}=0.$$

By Lemma 2.5 we obtain

$$\lim_{h\to 0\pm 0}\frac{Q'(h)}{P'(h)}=\lim_{h\to 0\pm 0}\frac{Q(h)}{P(h)}=\mp\frac{3\sqrt{3\pi}}{8}.$$

(ii) At $(h, P, Q)=(-\frac{4}{3}, 2, 4)$, the linear part of (6.10) is given by

$$\begin{pmatrix} 4 & 0 & 0 \\ -2 & -4 & 0 \\ -4 & 0 & -4 \end{pmatrix} \tag{6.12}$$

As such $(h, P(h), Q(h))$ belongs to the 1-dimensional unstable manifold at $(-\frac{4}{3}, 2, 4)$ and calculating the eigenspace belonging to the eigenvalue 4, one obtains that $\lim_{h\to 4/3+0} P'(h)=-\frac{1}{4}$, while $\lim_{h\to -4/3+0} Q'(h)=\frac{1}{2}$.

Similarly at $(h, P, Q)=(\frac{4}{3}, -2, 4)$ one obtains that $\lim_{h\to 4/3-0} P'(h)=-\frac{1}{4}$, while $\lim_{h\to 4/3-0} Q'(h)=\frac{1}{2}$. ∎

By using the results in Lemma 2.5, Theorem 5.2 and Lemma 6.3, we can draw the curves $v=P(h)$ and $v=Q(h)$ in a (h, v) plane (see Fig. 6).

As we mentioned at the end of Section 2, we will study the shape of the curve Ω, defined in PQ-plane by

$$Q=\tilde{Q}(P)=Q(h(P)), \tag{6.13}$$

where $h=h(P)$ is the inverse function of $P=P(h)$ for $h\in(-\frac{4}{3}, 0)\cup(0, \frac{4}{3})$. Knowing from (5.1) that the two branches of the curve Ω are symmetric

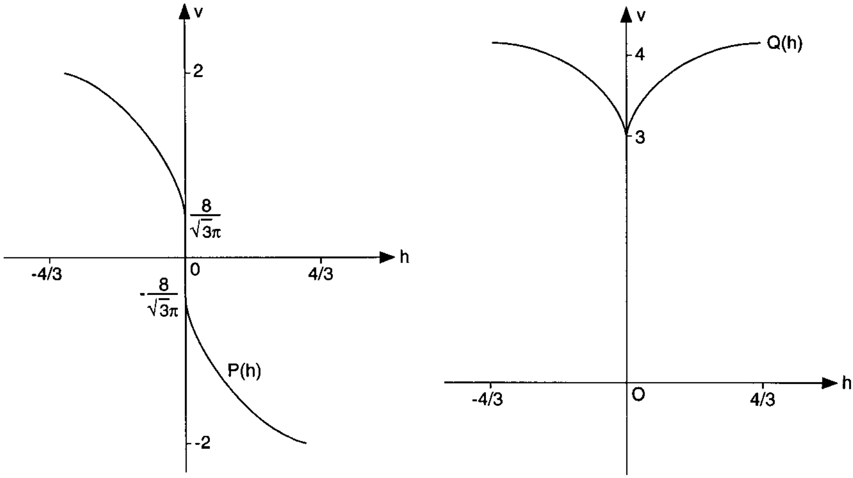

FIG. 6. Graphs of $P(h)$ and $Q(h)$.

with respect to the Q-axis, we only need to consider one branch γ_1 corresponding to $h \in (-\frac{4}{3}, 0)$. It is obvious that

$$\frac{d\tilde{Q}}{dP} = \frac{Q'(h)}{P'(h)}\bigg|_{h=h(P)}, \tag{6.14}$$

$$\frac{d^2\tilde{Q}}{dP^2} = \frac{Q''(h)\,P'(h) - P''(h)\,Q'(h)}{(P'(h))^3}\bigg|_{h=h(P)}. \tag{6.15}$$

THEOREM 6.4 *On each branch of the curve $\Omega : Q = \tilde{Q}(P)$ there is at least one inflection point* (*i.e. the point at which $d^2\tilde{Q}/dP^2 = 0$*).

Proof. We consider the branch γ_1 of Ω which joins the point $B_1(8/\sqrt{3\pi}, 3)$ to the point $A_1(2, 4)$. The slope, of the straight line segment from B_1 to A_1 is $(2 - 8/\sqrt{3\pi})^{-1} < 27/14 < 2$, since $\sqrt{3\pi} > 5.4$. Knowing from Lemma 6.3 that the slope of γ_1 at B_1 is $3\sqrt{3\pi/8} \approx 2.04$ and the slope at A_1 is 2, we conclude that γ_1 need to have at least one inflection point in the interior. ∎

7. THE UNIQUENESS OF THE INFLECTION POINT

From (6.9) we have that

$$\begin{cases} D(h)\,P'' = 1 + \frac{3}{4}hP - Q + \frac{3}{4}P^2 + (-3h^2 + 4 + \frac{3}{2}hP - \frac{2}{3}Q)\,P' \\ \qquad -(h + \frac{2}{3}P)\,Q', \\ D(h)\,Q'' = -3h - 3P + \frac{3}{2}hQ + \frac{3}{4}PQ + (-3h + \frac{3}{4}hQ)\,P' \\ \qquad +(-\frac{21}{8}h^2 + 4 + \frac{3}{4}hP - \frac{4}{3}Q)\,Q'. \end{cases}$$

Then we obtain

$$D(h)(Q''P' - P''Q') = A_{10}P' + A_{01}Q' + A_{20}P'^2 + A_{11}P'Q' + A_{02}Q'^2, \qquad (7.1)$$

where

$$A_{10} = -3h - 3P + \tfrac{3}{2}hQ + \tfrac{3}{4}PQ,$$

$$A_{01} = -(1 + \tfrac{3}{4}hP - Q + \tfrac{3}{4}P^2),$$

$$A_{20} = -3h + \tfrac{3}{4}hQ,$$

$$A_{11} = \tfrac{3}{8}h^2 - \tfrac{3}{4}hP - \tfrac{2}{3}Q,$$

$$A_{02} = h + \tfrac{2}{3}P.$$

Multiplying (7.1) by $D^2(h)$, substituting (6.9) into the right hand side, and then deleting $D(h)$ in both sides, we obtain

$$D^2(h)(Q''P' - P''Q') = \tfrac{1}{288}U(h, P, Q), \qquad (7.2)$$

where

$$\begin{aligned} U(h, P, Q) = {} & -54h^3P - 216h^2P^2 - 216hP^3 \\ & + (-576 + 288h^2 + 27h^3P + 135h^2P^2)\,Q \\ & + (384 - 144h^2 + 48hP)\,Q^2 - 64Q^3. \end{aligned} \qquad (7.3)$$

Since

$$\begin{aligned} \frac{\partial U}{\partial Q} = {} & 192(Q-1)(3-Q) + 9h^2(15P^2 - 32Q + 32) \\ & + 27h^3P + 96hPQ < 0 \end{aligned} \qquad (7.4)$$

for (h, P, Q) in the region

$$-\frac{4}{3}<h<0, \qquad \frac{8}{\sqrt{3}\pi}<P<2, \qquad 3<Q<4, \tag{7.5}$$

$U(h, P, Q)=0$ determines a surface $S_U: Q=\varphi(h, P)$ for (h, P) satisfying (7.5). Note that

$$U(0, P, Q)=-64Q(Q-3)^2,$$
$$U(-\tfrac{4}{3}, P, Q)=16(2P-Q)(3P+2Q-2)^2,$$

hence in region (7.5) $S_U \cap \{h=0\}$ is the straight line $\{Q=3\}$ and $S_U \cap \{h=-\frac{4}{3}\}$ is the straight line $\{Q=2P\}$. It is clear that both points $A(-\frac{4}{3}, 2, 4)$ and $B(0, 8/\sqrt{3}\pi, 3)$ are on (the boundary of) S_U.

By Lemma 2.5 we know that the curve Ω defined by (6.13) is the projection on (P, Q)-plane of an orbit of system (6.10); this orbit is the unstable manifold of (6.10) at the singularity A (see the linear part (6.12)), and it has B as its ω-limit; let us denote this 3-dimensional orbit by η_{AB}. By using (6.15) and (7.1) we conclude that proving that Ω has a unique inflection point, is equivalent to show that the orbit η_{AB} intersects the surface S_U at only one point. For this purpose, we first study the subset of S_U, at which the vector field (6.10) is tangent to S_U. This subset $G \subset S_U$ is given by

$$\begin{cases} U(h, P, Q)=0, \\ \dfrac{\partial U}{\partial h}\dot{h}+\dfrac{\partial U}{\partial P}\dot{P}+\dfrac{\partial U}{\partial Q}\dot{Q}=0, \end{cases} \tag{7.6}$$

where $(\dot{h}, \dot{P}, \dot{Q})$ is defined by (6.10).

LEMMA 7.1. (i) *$G=\sigma_1 \cup \sigma_2$, where $\sigma_i=S_U \cap S_i$, $S_i=\{(h, P, Q) \mid (h, P) \in \sigma'_i\}$, $i=1, 2$; σ'_1 has the equation*

$$4P^2+3hP-8=0, \tag{7.7}$$

defining a curve in the planar region

$$-\frac{4}{3}<h<0, \qquad \frac{8}{\sqrt{3}\pi}<P<2, \tag{7.8}$$

joining the point $(h, P)=(-\frac{4}{3}, 2)$ to the point $(h, P) \doteq (-0.146481, 8/\sqrt{3}\pi)$; σ'_2 can be expressed as a monotonic function $h=H(P)$, joining the point $(h, P) \doteq (-0.48797, 2)$ to the point $(h, P) \doteq (-0.122216, 8/\sqrt{3}\pi)$. σ'_1 and σ'_2 have no intersection in (7.8).

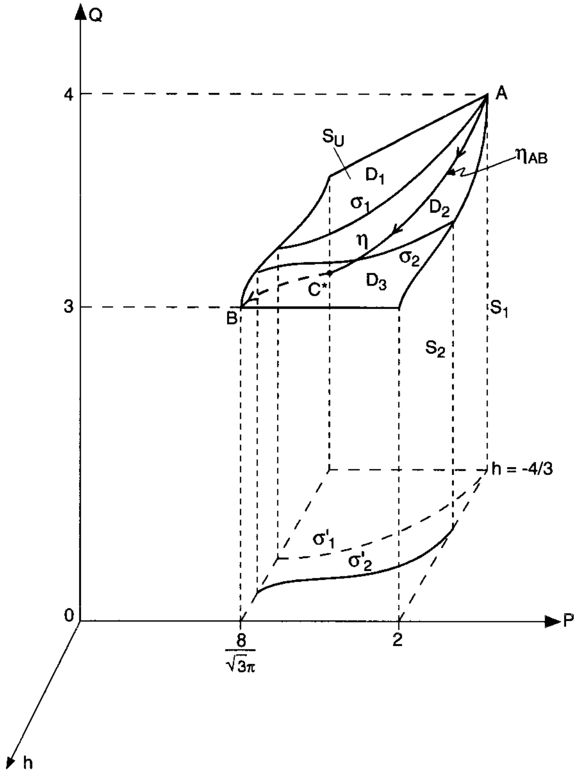

FIG. 7. 3-dimensional study of η_{AB}.

(ii) *S_U is divided by σ_1 and σ_2 into three disjoint open sets D_1, D_2 and D_3. On $D_1 \cup D_3$ (resp. D_2) the vector field* (6.10) *is pointing downwards (resp. upwards) relative to S_U. (S_i, σ_i, σ'_i $(i=1, 2)$ and D_k $(k=1, 2, 3)$ are illustrated in Fig.* 7 *forgetting the part of S_i above σ_i).*

Proof. See Appendices 1, 2, 3. ∎

LEMMA 7.2. *On the surface S_1 and above (resp. on or below) σ_1 we have $\dot{P} < 0$ (resp. $\dot{P} = 0$ or $\dot{P} > 0$).*

Proof. Using (7.7) we know that σ_1 is given by

$$\begin{cases} U(h, P, Q) = 0, \\ h = 4(2 - P^2)/3P, \end{cases}$$

where $U(h, P, Q)$ is defined in (7.3). Substituting the second equation into the first we have

$$\left(Q-\frac{P^2+4}{2}\right)[2P^2Q^2+(11P^4-44P^2+32)Q+2(P^2-2)(P^2+4)]=0. \tag{7.9}$$

Note that $11P^4-44P^2+32 \geqslant -12$ and it is not difficult to see that for (P, Q) satisfying (7.5) the expression in between brackets in (7.9) is different from zero. Hence the equation of σ_1 is

$$\begin{cases} Q=\dfrac{P^2+4}{2} \\ h=\dfrac{4(2-P^2)}{3P}. \end{cases} \tag{7.10}$$

A direct calculation shows that (7.10) is exactly the solution of

$$\begin{cases} 4P^2+3hP-8=0, \\ \dot{P}=0, \end{cases}$$

where $\dot{P}$ is given by (6.10). Notice that on S_1 we have

$$\frac{\partial \dot{P}}{\partial Q}=\left(-h-\frac{2}{3}P\right)\bigg|_{h=4(2-P^2)/3P}=\frac{2(P^2-4)}{3P}<0$$

for $P<2$, implying the desired conclusion. ∎

LEMMA 7.3. *Along and below σ_2 the vector field* (6.10) *is transverse to the surface S_2 pointing in the direction of* $\{h=0\}$.

Proof. See Appendix 4. ∎

LEMMA 7.4. *Near the point A the orbit η_{AB} is situated above the surface S_U.*

Proof. From Lemma 6.3 we know that the first order approximation of η_{AB} near A is $(h, P, Q)=(-\frac{4}{3}+\bar{h}, 2-\frac{1}{4}\bar{h}, 4-\frac{1}{2}\bar{h})$ with small $\bar{h}$. To find the second order approximation, we let

$$\begin{cases} h=-\frac{4}{3}+\bar{h}, \\ P=2-\frac{1}{4}\bar{h}+\alpha\bar{h}^2+O(\bar{h}^3), \\ Q=4-\frac{1}{2}\bar{h}+\beta\bar{h}^2+O(\bar{h}^3). \end{cases} \tag{7.11}$$

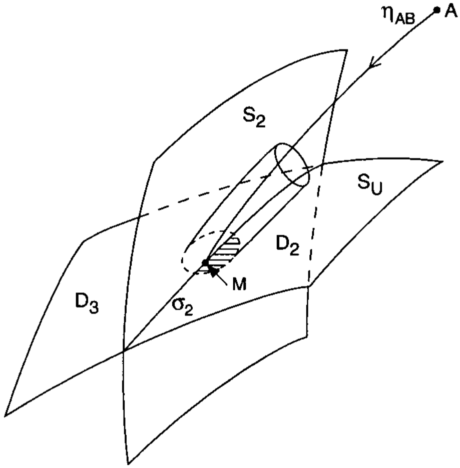

FIG. 8. Obstruction to possible position of η_{AB}.

Substituting (7.11) into (6.10), we find $\alpha = -\frac{25}{576}$, $\beta = -\frac{19}{288}$. Taking $(\hat{h}, \hat{P}, \hat{Q}) = (-\frac{4}{3} + \bar{h}, 2 - \frac{1}{4}\bar{h} - \frac{25}{576}\bar{h}^2 + O(\bar{h}^3), 4 - \frac{1}{2}\bar{h} - \frac{19}{288}\bar{h}^2 + O(\bar{h}^3))$, we have

$$U(\hat{h}, \hat{P}, \hat{Q}) = \bar{h}^2(-48 + O(\bar{h})).$$

Hence $U(\hat{h}, \hat{P}, \hat{Q}) < 0$ for $0 < \bar{h} \ll 1$. On the other hand, (7.4) shows that $\partial U/\partial Q < 0$ for (h, P, Q) in (7.5). Thus η_{AB} is above the surface S_U near the point A. ∎

LEMMA 7.5. $\eta_{AB} \cap (\sigma_1 \cup \sigma_2) = \varnothing$.

Proof. By Lemma 7.2 and Theorem 5.3, η_{AB} certainly can not meet σ_1. If η_{AB} meets σ_2 at some point M, by Lemma 7.3 η_{AB} must be transverse to S_2 at M. By using the continuous property of solutions upon the initial conditions, we can find a tubular neighborhood of η_{AB} near M, such that the flow inside the tube goes in the same direction as η_{AB} does. The tube must intersect D_2 (see Fig. 8), this contradicts Lemma 7.1(ii). ∎

THEOREM 7.6. *On each branch of the curve Ω, defined by* (6.13) *and related to* (2.15), *there is a unique inflection point, at which $d^2\tilde{Q}/dP^2 = 0$ and $d^3\tilde{Q}/dP^3 \neq 0$.*

Proof. By (6.15) and (7.2) we have

$$\frac{d^2\tilde{Q}}{dP^2} = \frac{1}{288} \left. \frac{U(h, P(h), Q(h))}{D^2(h)P'^3(h)} \right|_{h = h(P)}. \tag{7.12}$$

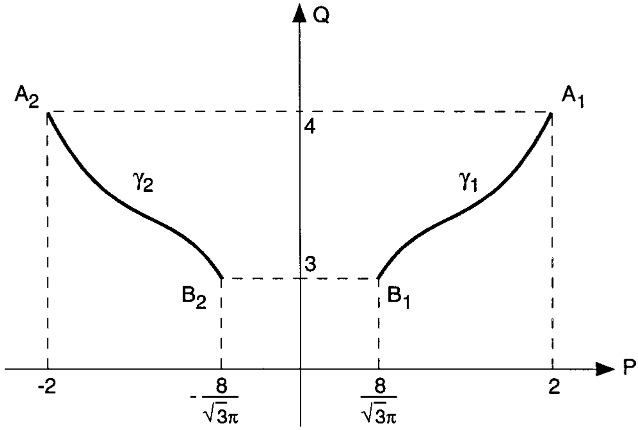

FIG. 9. Graph of $Q(P)$.

On the other hand, by (7.4) we know that $\partial U/\partial Q<0$. Hence the fact that η_{AB} is above S_U near the point A (Lemma 7.4) means that $d^2\tilde{Q}/dP^2>0$ for $0<2-P\ll 1$. By Theorem 6.4, η_{AB} must meet S_U in region (7.5).

Let us follow the position of η_{AB} following an increasing t (or h) (see Figure 7). Near the point A, it is above S_U, then, by Lemma 7.1(ii) and Lemma 7.5 it could meet S_U at a point either on D_1 or on D_3. If this point is on D_1, then below S_U η_{AB} can neither go through S_1 (by Lemma 7.2), nor meet S_U again (by Lemma 7.1(ii)), hence there is no way to go to the point B. Thus, the only possibility is that η_{AB} goes through S_2 at a point above σ_2, then goes through S_U at a point on D_3. Below S_U, it can neither meet S_2 again (by Lemma 7.3), nor intersect S_U again (by Lemma 7.1(ii)), hence can only go to the point B. This finishes the proof of the uniqueness of the inflection point.

Now we suppose that η_{AB} intersects S_U at the point $C^*(h^*, P^*, Q^*)$, then $d^2\tilde{Q}/dP^2|_{P*}=0$. From (7.12) we have that

$$\frac{d^3\tilde{Q}}{dP^3}=\frac{1}{288}\frac{d}{dh}\left(\frac{U}{D^2P'^3}\right)\frac{dh}{dP}=\frac{(DP')\dfrac{dU}{dh}-(3DP''+2D'P')U}{D^3P'^5}.$$

Hence, if $d^3\tilde{Q}/dP^3|_{P*}$ is also equal to zero, then (h^*, P^*, Q^*) is a solution of the equations

$$\begin{cases} U(h, P, Q)=0, \\ \dfrac{dU}{dt}=\dfrac{\partial U}{\partial h}\dot{h}+\dfrac{\partial U}{\partial P}\dot{P}+\dfrac{\partial U}{\partial Q}\dot{Q}=0. \end{cases}$$

Comparing with (7.6), we conclude that $(h^*, P^*, Q^*) \in \sigma_1 \cup \sigma_2$, contradicting Lemma 7.5. ▮

Remark 7.7. From the previous proof we know that $d^2\tilde{Q}/dP^2 < 0$ for $0 < P - 8/\sqrt{3}\pi \ll 1$, $d^2\tilde{Q}/dP^2 > 0$ for $0 < 2 - P \ll 1$.

The curve Ω is illustrated in Fig. 9.

8. THE NUMBER OF LIMIT CYCLES AND LIMIT CYCLE BIFURCATIONS

As we discussed in Sections 1 and 2, under non-conservative perturbations, the number of limit cycles of $(2.4)_\delta$ for $0 < \delta \ll 1$ is equal to the number of zeros of the Abelian integral

$$I(h) = (v_2 - 4v_3)\, I_0(h) + (v_1 + \tfrac{1}{2}v_2)\, I_1(h) + v_3 I_2(h) \tag{8.1}$$

(for h in any compact interval in $(-\frac{4}{3}, 0) \cup (0, \frac{4}{3})$). This is the same as for the function

$$M(h) = (v_2 - 4v_3) + (v_1 + \tfrac{1}{2}v_2)\, P(h) + v_3\, Q(h),$$

or

$$\tilde{M}(P) = (v_2 - 4v_3) + (v_1 + \tfrac{1}{2}v_2) P + v_3 \tilde{Q}(P) \tag{8.2}$$

where $\tilde{Q}(P) = Q(h(P))$ as before.

It is easy to prove the following.

LEMMA 8.1. *Suppose $h \in (-\frac{4}{3}, 0) \cup (0, \frac{4}{3})$. The following three conditions are necessary and sufficient for $(2.4)_\delta$ to have a limit cycle near Γ_h for $0 < \delta \ll 1$; it is respectively hyperbolic, with multiplicity* 2, *or with multiplicity* 3:

(1) $I(h) = 0,\ I'(h) \neq 0;$

(2) $I(h) = I'(h) = 0,\ I''(h) \neq 0;$

(3) $I(h) = I'(h) = I''(h) = 0,\ I'''(h) \neq 0.$

These conditions are equivalent to respectively the following three conditions:

(i) $\tilde{M}(P) = 0,\ \tilde{M}'(P) \neq 0;$

(ii) $\tilde{M}(P) = \tilde{M}'(P) = 0,\ \tilde{M}''(P) \neq 0;$

(iii) $\tilde{M}(P) = \tilde{M}'(P) = \tilde{M}''(P) = 0,\ \tilde{M}'''(P) \neq 0.$ ▮

LEMMA 8.2. *If the parameter* $\nu=(\nu_1, \nu_2, \nu_3)$ *is in the faces* $\{\nu_1=\pm M, |\nu_2|\leqslant M, 0\leqslant \nu_3\leqslant 1\}$ *or* $\{\nu_2=\pm M, |\nu_1|\leqslant M, 0\leqslant \nu_3\leqslant 1\}$ *with* M *large, then the corresponding system* $(2.4)_\delta$ *has no multiple limit cycles.*

Proof. By Lemma 8.1, a necessary condition for the existence of a multiple limit cycle is $\tilde{M}(P)=0$ and

$$\tilde{M}'(P)=(\nu_1+\tfrac{1}{2}\nu_2)+\nu_3\tilde{Q}'(P)=0. \tag{8.3}$$

By Lemma 6.3, $\tilde{Q}'(P)=Q'(h)/P'(h)|_{h=h(P)}$ is bounded for all P. Hence (8.3) can not be satisfied if $|\nu_3|\leqslant 1$, $\nu_1=\pm M$ and $|\nu_2|\leqslant M$ with M large. On the other hand, by using (8.2) and (8.3) we know that $\tilde{M}(P)=\tilde{M}'(P)=0$ implies

$$\nu_2-4\nu_3+\nu_3(\tilde{Q}(P)-P\tilde{Q}'(P))=0.$$

Since P, $\tilde{Q}(P)$ and $\tilde{Q}'(P)$ are all bounded, the above expression can not be satisfied on the face $\{\nu_2=\pm M, |\nu_1|\leqslant M, 0\leqslant\nu_3\leqslant 1\}$ with M large. ∎

Thus, to consider the multiple limit cycle bifurcations, we only need to study systems on the face $\{\nu_3=1, |\nu_1|\leqslant M, |\nu_2|\leqslant M$, with M large$\}$. ∎

Taking $\nu_3=1$, we have

$$\tilde{M}(P)=(\nu_2-4)+(\nu_1+\tfrac{1}{2}\nu_2)\,P+\tilde{Q}(P). \tag{8.4}$$

We rewrite $\tilde{M}(P)=0$ as

$$\nu_1=\varphi(P)\,\nu_2+\Psi(P), \tag{8.5}$$

where

$$\begin{cases} \varphi(P)=-\left(\dfrac{1}{2}+\dfrac{1}{P}\right) \\ \Psi(P)=\dfrac{1}{P}(4-\tilde{Q}(P)). \end{cases} \tag{8.6}$$

We note that for $h\in(-\frac{4}{3},0)\cup(0,\frac{4}{3})$, we have $|P|>0$, and $\varphi'(P)=1/P^2\neq 0$. Hence we can take φ as new parameter instead of P, and write (8.5) as

$$\nu_1=\varphi\nu_2+\tilde{\Psi}(\varphi), \tag{8.7}$$

where

$$\tilde{\Psi}(\varphi)=\Psi(P(\varphi)), \tag{8.8}$$

$P(\varphi)$ is the inverse function of $\varphi(P)$. We note that $\varphi\in((-\sqrt{3}\pi-4)/8, -1)\cup(0, (\sqrt{3}\pi-4)/8)$.

LEMMA 8.3. *System* $(2.4)_\delta$ $(0<\delta\ll 1)$ *has a limit cycle near* Γ_h *with multiplicity* 2 *or* 3, *if and only if respectively*

$$\begin{cases} v_1=\varphi v_2+\tilde{\Psi}(\varphi), \\ v_2+\tilde{\Psi}'(\varphi)=0, \\ \tilde{\Psi}''(\varphi)\neq 0; \end{cases} \tag{8.9}$$

or

$$\begin{cases} v_1=\varphi v_2+\tilde{\Psi}(\varphi), \\ v_2+\tilde{\Psi}'(\varphi)=0, \\ \tilde{\Psi}''(\varphi)=0, \qquad \tilde{\Psi}'''(\varphi)\neq 0 \end{cases} \tag{8.10}$$

is satisfied.

Proof. By using (8.6) and (8.8) we have

$$\begin{aligned} \tilde{\Psi}'(\varphi)&=-\tilde{Q}'(P)P-(4-\tilde{Q}(P)), \\ \tilde{\Psi}''(\varphi)&=-\tilde{Q}''(P)P^3, \\ \tilde{\Psi}'''(\varphi)&=-(\tilde{Q}'''(P)P+3\tilde{Q}''(P))P^4, \end{aligned}$$

where $P=P(\varphi)$. Hence

$$\begin{aligned} &v_1=\varphi v_2+\tilde{\Psi}(\varphi), \qquad v_2+\tilde{\Psi}'(\varphi)=0, \qquad \tilde{\Psi}''(\varphi)\neq 0 \\ \Leftrightarrow\ &\tilde{M}(P)=0, \qquad v_2-4-P\tilde{Q}'(P)+\tilde{Q}(P)=0, \qquad \tilde{Q}''(P)\neq 0 \\ \Leftrightarrow\ &\tilde{M}(P)=0, \qquad \tilde{M}'(P)=0, \qquad \tilde{M}''(P)\neq 0. \end{aligned}$$

By Lemma 8.1 (taking $v_3=1$), the first part of the lemma is true. The proof for the second part is similar. ∎

THEOREM 8.4. *The double limit cycle bifurcation curve of* $(2.4)_\delta$ *in* v_1v_2*-plane* $(v_3=1)$ *for* $0<\delta\ll 1$ *is the envelope of the family of lines*

$$v_1=cv_2+\tilde{\Psi}(c), \tag{8.11}$$

where the function $\tilde{\Psi}$ *is defined in* (8.8), *the parameter* $c\in((-\sqrt{3}\pi-4)/8, -1)\cup(0, (\sqrt{3}\pi-4)/8)$, *corresponding to the two branchs of the envelope* $DC_1\cup DC_2$. *For* $i=1, 2$, *each branch* DC_i *consists of two pieces of curves, one is concave while the other is convex. They are tangent to* H_i *and* L_i *at* $\{DH_i\}$ *and* $\{DL_i\}$ *respectively on one side, and tangent to each other at*

$\{TC_i\}$ on the other side. The point $\{TC_i\}$ corresponds to a triple limit cycle bifurcation. In the cuspidal region, formed by the two pieces of DC_i, H_i and L_i, the corresponding systems have exactly 3 *limit cycles surrounding the singular point C_i. See Fig.* 2 *for more details.*

Proof. By Lemma 8.3, (8.9) is the condition for existence of double limit cycles. If we consider the problem in $v_1 v_2$-plane, taking φ as parameter, then we can write the first equation of (8.9) into the form

$$v_1 = v_2 \frac{dv_1}{dv_2} + \tilde{\Psi}\left(\frac{dv_1}{dv_2}\right), \tag{8.12}$$

which is the Clairaut equation. Hence, (8.11) is just the general solution of (8.12), and (8.9) gives the singular solution of (8.12). Therefore, the graph of (8.9) is just the envelope of the family of lines (8.11). The graph is divided into two parts by the point $\{TC_i\}$ at which $\tilde{\Psi}''(\varphi) = 0$ and $\tilde{\Psi}'''(\varphi) \neq 0$ (Theorem 7.6). By the second part of Lemma 8.3, $\{TC_i\}$ corresponds to a triple limit cycle bifurcation. Since in the two parts of DC_i, $\tilde{\Psi}''(\varphi)$ have different signs, and the envelope has the same concave or convex property as the curve $-\tilde{\Psi}(\varphi)$ has, see [A] for example, one of the two pieces of DC_i must be concave while the other one is convex. ▮

Now we can describe the number of limit cycles for system $(2.4)_\delta^*$, corresponding to a given (v_1^*, v_2^*) $(v_3 = 1)$, in the following way: $(2.4)_\delta^*$ has at least one limit cycle if and only if (v_1^*, v_2^*) is located in the region covered by the family of lines (8.11), denoted by $\{\ell_c \mid c \in ((-\sqrt{3\pi - 4})/8, -1) \cup (0, (\sqrt{3\pi - 4})/8)\}$. And the number of limit cycles of $(2.4)_\delta^*$ equals the number of different lines $\subset \{\ell_c\}$, passing through the same point (v_1^*, v_2^*). This is just the number of lines which pass through the point (v_1^*, v_2^*) and are tangent to the curve DC_i, because DC_i is the evelope of the family of lines $\{\ell_c\}$ (see Theorem 8.4).

It is clear that the region mentioned above can be constructed by moving the line ℓ_c from the position of H_i to the position of L_i, changing c from -1 to $(-\sqrt{3\pi - 4})/8$, or from 0 to $(\sqrt{3\pi - 4})/8$, depending on $i = 1$ or 2. In Fig. 10 we give the picture for $i = 2$. In Fig. 11, we indicate the corresponding number of limit cycles for each subregion. It is just the number described geometrically above.

Remark 8.5. To draw the Figures 10 and 11, it is important to know that the triple limit cycle bifurcation point is unique (for each branch DC_i) which is guaranteed by Theorem 7.6. Otherwise, some more complicated situations might occur. Fig. 12 gives a possible bifurcation diagram with three cuspital points being compatible with all information we get, except for Theorem 7.6.

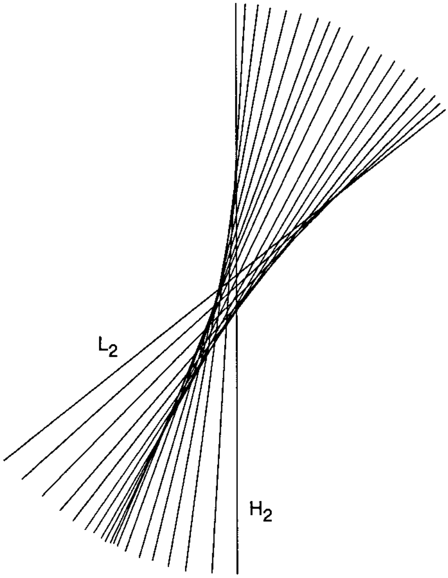

FIG. 10. Bifurcation curve as envelope of straight lines.

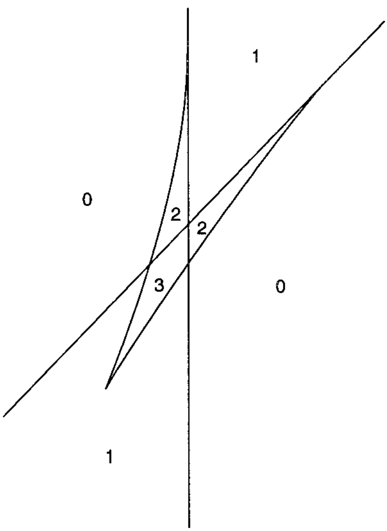

FIG. 11. A region in parameter space with three limit cycles.

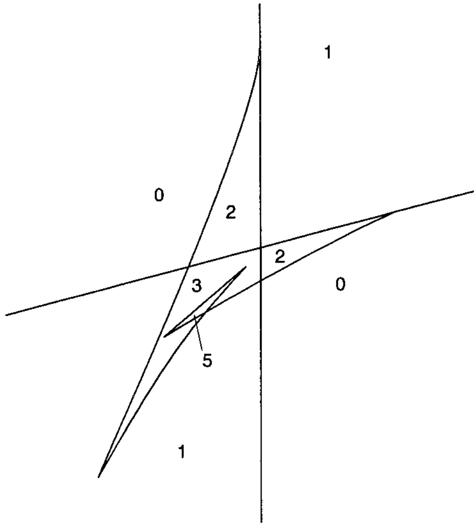

FIG. 12. Possible complications in bifurcation diagram.

9. THE PROOF OF THEOREM 1.2

The conclusions (i) and (ii) of Theorem 1.2 are proved in Lemma 2.4 and Lemma 2.1 respectively. Conclusion (iii) concerns the bifurcation diagram and corresponding phase portraits. The possible bifurcations for $(2.4)_\delta$ have three types: Hopf, heteroclinic loop, and multiple limit cycle bifurcations. We have discussed them completely in Section 3, 4 and 8 respectively.

The only statement which remains to be shown deals with the relative position of the two sets of multiple limit cycle bifurcation curves DC_1 and DC_2. We need to check whether it is possible or not to have an intersection for the two cuspidal regions formed by DC_1 and DC_2, or at least, to have an intersection for each cuspital region with the "region a" of Fig. 2? We will prove that the answer is negative.

In fact, we consider the number of zeros of (8.4) with respect to $P \in (-2, -(8/\sqrt{3}\pi)) \cup (8/\sqrt{3}\pi, 2)$, for different given (v_1, v_2). This is equivalent to finding the number of intersection points of the curve

$$\Omega : Q = \tilde{Q}(P), \tag{9.1}$$

and the straight line

$$\ell : Q = -(v_1 + \tfrac{1}{2}v_2)P - (v_2 - 4). \tag{9.2}$$

Hence, we go back to Fig. 9 where the two branches γ_1 and γ_2 of the curve Ω are illustrated. By Lemma 2.5, Theorem 5.3, Lemma 6.3 and Theorem 7.6, we know that each γ_i is monotonic with unique inflection point, and the slopes of γ_i satisfy

$$|\tilde{Q}'(P^*)| \leqslant |\tilde{Q}'(P)| \leqslant \left|\tilde{Q}'\left(\pm\frac{8}{\sqrt{3\pi}}\right)\right| = \frac{3\sqrt{3}}{8}\pi \approx 2.04 \tag{9.3}$$

where (P^*, Q^*) is the inflection point.

We give the corresponding relations between the bifurcation diagram in Fig. 2 and the geometric positions of the line ℓ with the curve γ_i in Fig. 9:

(1) The Hopf (resp. heteroclinic loop) bifurcation curve H_i (resp. L_i) consists of all such (v_1, v_2), for which the line (9.2) passes through the end point A_i (resp. B_i) in Fig. 9; the codimension 2 bifurcation point $\{DH_i\}$ (resp. $\{DL_i\}$) in Fig. 2 has the coordinate (v_1, v_2), for which the line (9.2) is tangent to γ_i at A_i (resp. B_i).

(2) If a line (9.2) crosses γ_i for $P \in (-2, -(8/\sqrt{3\pi})) \cup (8/\sqrt{3\pi}, 2)$, then the corresponding system has at least one limit cycle surrounding the singular point C_i.

(3) The double limit cycle bifurcation curve DC_i consists of all such (v_1, v_2), for which the line (9.2) is tangent to γ_i at any point of γ_i except for the points A_i, B_i and the inflection point.

(4) The triple limit cycle bifurcation point $\{TC_i\}$ has the coordinate (v_1, v_2), for which the line (9.2) is tangent to γ_i at the unique inflection point.

By the uniqueness of the inflection point (Theorem 7.6), any line (9.2) can cross γ_1 (or γ_2) at most at 3 points. This implies that around one singular point there are at most 3 limit cycles.

We prove now that any tangent line of γ_i never meet another branch γ_j, $j \neq i$, $(i, j = 1, 2)$, which implies that the bifurcation curve DC_i in Fig. 2 has no intersection with H_j and L_j, $j \neq i$. This induces the exact location of the two cuspidal regions in Fig. 2, and proves that $(1, 1)$ is the unique possible configuration of limit cycles surrounding the two foci simultaneously.

Since the tangent line at the inflection point A^* of γ_1 has minimum slope, if this tangent line crosses γ_2, then we must find two points A_1^* and A_2^* on γ_1 at left and right side of A^* respectively, such that the tangent lines of γ_1 at A_1^* and A_2^* pass through the same point $(P, Q) = (0, 3)$, see Fig. 13. This implies that the equation

$$\frac{Q'(h)}{P'(h)} - \frac{Q(h) - 3}{P(h)} = 0 \tag{9.4}$$

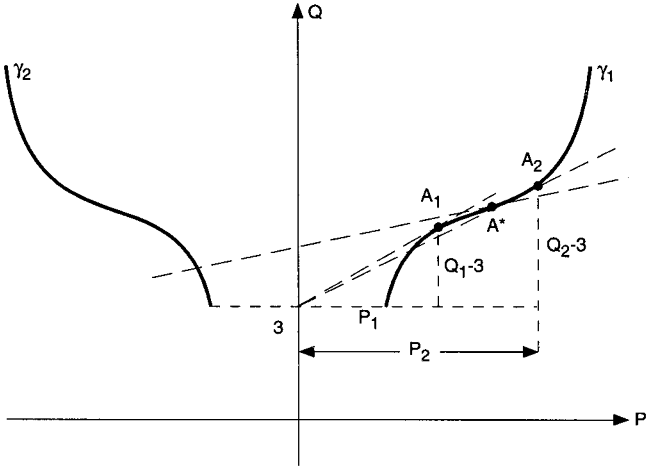

FIG. 13. How to position TC_1 and TC_1 in parameter space?

has at least two solutions for $h \in (-\frac{4}{3}, 0)$, where $P(h)$ and $Q(h)$ are solutions of the differential equation (6.9). We will prove that this is impossible.

In fact, substituting (6.9) into (9.4) and factorizating, we have

$$(8Q + 3hP - 24)(-hQ + 2P + h) = 0. \tag{9.5}$$

Since we consider h, P, Q in the region

$$-\frac{4}{3} < h < 0, \qquad \frac{8}{\sqrt{3\pi}} < P < 2, \qquad 3 < Q < 4, \tag{9.6}$$

it is obvious that $-hQ + 2P + h > 0$. Hence (9.5) is equivalent to

$$8Q + 3hP - 24 = 0. \tag{9.7}$$

We suppose that (9.4) has two solutions for $h \in (-\frac{4}{3}, 0)$, which implies that in (h, P, Q)-space the surface S defined by (9.7) has at least two intersection points B_1 and B_2 with a trajectory T of the system (6.10) (see Fig. 14).

We denote the normal vector of S at the point $(h, P, Q) \in S$ by $\mathbf{n} = (3P, 3h, 8)$, and the tangent vector of system (6.10) at any point (h, P, Q) by $\mathbf{v}$.

We first prove that there is at least one point on S, at which $\mathbf{n} \cdot \mathbf{v} = 0$. In fact, if both $\mathbf{n}_1 \cdot \mathbf{v}_1$ and $\mathbf{n}_2 \cdot \mathbf{v}_2$ are non-zero, then they must have different signs. By the smoothness of S and (6.10), moving a point on S from B_1

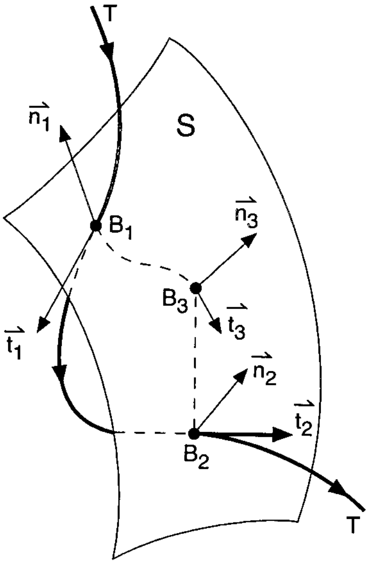

FIG. 14. Studying η_{AB} in 3-space.

to B_2, we must find a point B_3 on S such that $\mathbf{n}_3 \cdot \mathbf{v}_3|_{B_3} = 0$. Hence, in any case the simultaneous equations

$$\begin{cases} 8Q + 3hP - 24 = 0, \\ \mathbf{n} \cdot \mathbf{v} = 0 \end{cases} \tag{9.8}$$

have at least one solution in the region (9.6).

Solving Q from the first equation of (9.8), substituting it into the second one, and using the expression of $\mathbf{v}$ in (6.10), we obtain

$$h(-4+3h)(4+3h)P = 0,$$

which obviously has no solution in the region (9.6).

At last, we need to show that the bifurcation diagram (Fig. 2) can be lifted to the case $0 < \delta \ll 1$. This essentially can be done by the implicit function theorem. But near the Hopf and heteroclinic bifurcations this is not enough. From (3.1) we know that along the Hopf bifurcation line H_i we have $\lim_{\delta \to 0} W_1/\delta \neq 0$. By using Theorem 2.5 of Chapter 3 in [CLW], results are valid in a fixed neighbourhood. When $W_1 = 0$ (at the bifurcation point DH_i) we have $\lim_{\delta \to 0} W_2/\delta \neq 0$. A similar argument now applies to provide the required results near DH_i in a uniform way. For heteroclinic loop bifurcations, from (4.14) and (4.15) we know that all $C_0(\lambda)$, $C_1(\lambda)$ and $C_2(\lambda)$ have the factor δ, hence here again we get the required results.

Thus, the proof of Theorem 1.2 is finished.

APPENDIX 1: THE PROOF OF LEMMA 7.1

We finish the calculations in the Appendices 1–4 by using "Mathematica". So we prefer not to include all intermediate results of the calculation, but focus on the computation process itself only including the most important steps and results. The information should suffice to check the computation.

Substituting (7.3) and (6.10) into the second equation of (7.6), we get the equation

$$\begin{aligned} U_1 = {} & 3456h^2 - 1944h^4 + 6912hP - 4320h^3P - 972h^5P \\ & - 2592h^2P^2 - 3888h^4P^2 - 3456hP^3 - 4860h^3P^3 - 1944h^2P^4 \\ & + (-4608 - 8640h^2 + 5508h^4 - 10944hP + 6480h^3P + 486h^5P \\ & + 3672h^2P^2 + 2187h^4P^2 + 1728hP^3 + 1215h^3P^3)\,Q \\ & + (7680 + 2880h^2 - 2268h^4 + 5376hP - 1512h^3P - 648h^2P^2)\,Q^2 \\ & - (3584 + 960hP)\,Q^3 + 512Q^4 = 0. \end{aligned}$$

Note that

$$U_1 = (8 + 18h^2 + 9hp - 8Q)\,U + 48D(h)\,V,$$

where $D(h)$ is the same as in (6.10), and

$$\begin{aligned} V(h, P, Q) = {} & -36h - 72P + 9hP^2 + 18P^3 + (6h + 60P - 9hP^2)\,Q \\ & + (6h - 16P)\,Q^2, \end{aligned} \tag{A.1}$$

hence (7.6) is equivalent to

$$U(h, P, Q) = 0, \qquad V(h, P, Q) = 0. \tag{A.2}$$

From $V(h, P, Q) = 0$ we obtain

$$Q = \Psi \pm (h, p) = \frac{R_1 \pm \sqrt{R_1^2 - 4R_0R_2}}{-2R_2}, \tag{A.3}$$

where $R_0 = 9(h+2P)(P^2-4) < 0$, $R_1 = 6h+60P-9hP^2 > 0$, and $R_2 = 6h-16P < 0$ for (h, P, Q) satisfying (7.5). Since $R_1/(-2R_2) < 3$, only $Q = \Psi_+(h, P)$ should be considered. But we remark here that if we eliminate Q from (A.2) to get an equation of h and P, we will also find the projection of the intersection of S_U and $\{Q = \Psi_-(h, P)\}$ onto the (h, P)-plane. We need to use this fact several times in the sequel.

Now (A.2) is equivalent to

$$U(h, P, Q) = 0, \qquad U(h, P, \Psi_+(h, P)) = 0. \tag{A.4}$$

A numerical computation indicates that the locus of the second equation of (A.4) in region (7.8) looks like Fig. 15: its intersection with the boundary of the rectangle (7.8) consists of 4 points: $A_1 = (-\frac{4}{3}, 2)$, $A_2 \doteq (-0.48797, 2)$, $B_1 \doteq (-0.146481, 8/\sqrt{3\pi})$, $B_2 \doteq (-0.122216, 8/\sqrt{3\pi})$.

For a precise study, we eliminate Q from (A.2), and obtain

$$\begin{cases} U(h, P, Q) = 0, \\ (4P^2 + 3hP - 8)\, W(h, P) = 0, \end{cases} \tag{A.5}$$

where

$$\begin{aligned} W(h, P) = {} & -8192h + 2560h^3 + 8192P - 9192h^2P + 352h^4P \\ & + 13312hP^2 - 5632h^3P^2 + 444h^5P^2 - 6144P^3 + 11008h^2P^3 \\ & - 1832h^4P^3 + 27h^6P^3 - 5376hP^4 + 3936h^3P^4 - 261h^5P^4 \\ & + 1024P^5 - 4224h^2P^5 + 468h^4P^5. \end{aligned} \tag{A.6}$$

It is clear that σ_1' has the equation (7.7) and joins the points A_1 and B_1.

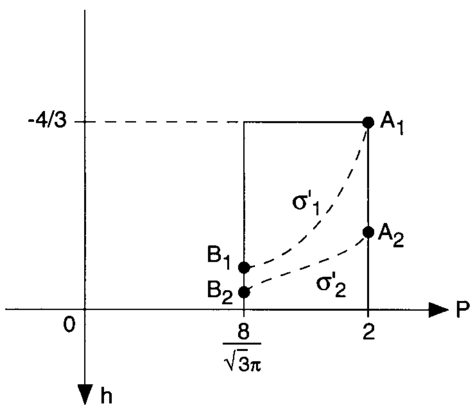

FIG. 15. Projection on a (P, h)-plane.

In Appendices 2 and 3 we will prove that the equation

$$W = 0, \qquad \frac{\partial W}{\partial h} = 0;$$

and

$$W = 0, \qquad \frac{\partial W}{\partial P} = 0$$

have a unique common solution $(h^*, P^*) \doteq (-0.256425, 1.56633)$. By this fact it is easy to find out that the locus of $W(h, P) = 0$ in the region (7.8) consists of two curves $\sigma_2' : h = H(P)$ and $\sigma_3' : H = \tilde{H}(P)$ with the following properties:

(1) $H'(P) < 0$, $\tilde{H}'(P) > 0$;

(2) σ_2' intersects σ_3' at (h^*, P^*);

(3) σ_2' joins the points A_2 and B_2, σ_3' joins the points $A_3 \doteq (-0.346851, 8/\sqrt{3}\pi)$ and $B_3 = (0, 2)$. (See Fig. 16.)

We remark that the property (1) can be obtained directly for $P \neq P^*$, hence $\lim_{P \to P^*} H'(P)$ exists (or infinity). Note that $W_h'(h^*, P^*) = W_P'(h^*, P^*) = 0$, $H'(P^*)$ satisfies

$$H'(P^*) = -\left.\frac{W''_{hP} H'(P) + W''_{PP}}{W''_{hh} H'(P) + W''_{hP}}\right|_{(h^*, P^*)}.$$

From this equality, we get $H'(P^*) \doteq -1.08753$, $\tilde{H}'(P^*) \doteq 0.85717$.

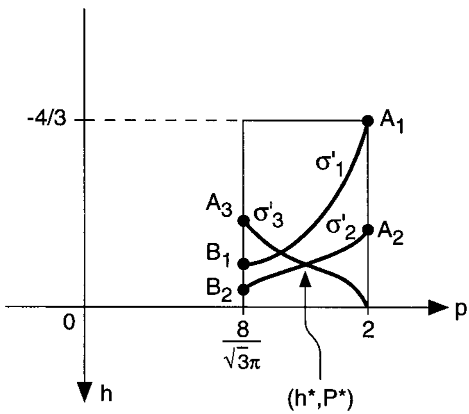

FIG. 16. Study of η_{AB} in (P, h)-plane.

TABLE I

$\mathbf{v}\cdot\mathbf{n}$ on S_U

	h	P	$Q \doteq$	$\frac{\partial U}{\partial h}$	$\frac{\partial U}{\partial p}$	$\frac{\partial U}{\partial h}$	$\mathbf{v}\cdot\mathbf{n}$	Remark
1	-1.1	1.8	3.61562	+	+	−	+	on D_1
2	-0.6	1.8	3.60072	−	+	−	−	on D_2
3	-0.2	1.8	3.46313	−	+	−	+	on D_3

It is easy to see that σ_3' comes from the intersection of S_U with $\{Q=\Psi_-(h, P)\}$ as we remarked before, hence it should not be considered.

To finish the proof of Lemma 7.1(i), we need to show that σ_1' and σ_2' have no intersection.

From (7.7) we get $h=(8-4P^2)/3P$, substituting it into $W(h, P)=0$, we obtain

$$(P^2-1)(P^2-2)^2\,(3P^2-7)(3P^4-32P^2+16)=0.$$

The only root of this equation for $8/\sqrt{3}\pi<P<2$ is $\hat{P}\doteq 1.52753$ which corresponds to the intersection of σ_3' with σ_1' (see Fig. 16).

To prove the conclusion (ii) of Lemma 7.1, it is sufficient to consider 3 points in D_1, D_2 and D_3 respectively, and to check the position of the vector field (6.10) relative to the normal direction of the surface S_U at each point (see Fig. 7).

For example, we list some results in Table I.

APPENDIX 2: THE STUDY OF $\partial W/\partial h$

We need to prove that the equations

$$W(h, P)=0, \qquad \frac{\partial W}{\partial h}(h, P)=0 \tag{A.7}$$

have only one solution $(h^*, P^*)\doteq(-0.256425, 1.56633)$ in the region

$$-0.48797<h<-0.122216, \qquad \frac{8}{\sqrt{3}\pi}<P<2. \tag{A.8}$$

Note that both equations in (A.7) are polynomials in h and P (see (A.6)). Computation permits to reduce (A.7) to the form

$$A(h)P+B(h)=0, \qquad Z(h)=0, \tag{A.9}$$

where A, B and Z are polynomials of h. We need to remark here that in reducing (A.7) to (A.9), we may multiply it by some polynomials, possibly causing some extra solutions.

In our case, we find two solutions of (A.9): $(h, P) = (h^*, P^*)$ and $(h, P) \doteq (-0.279302, 1.58843)$. Only the first one satisfies (A.7).

APPENDIX 3: THE STUDY OF $\partial W/\partial P$

By using the same procedure as in Appendix 2, we can reduce the equations

$$W(h, P) = 0, \qquad \frac{dW}{dP}(h, P) = 0 \tag{A.10}$$

to the form (A.9), and find the unique solution (h^*, P^*) in the region (A.8).

APPENDIX 4: THE PROOF OF LEMMA 7.3

First, we study the subset of S_2 at which the vector field (6.10) is tangent to S_2. This subset $\tau \subset S_2$ is given by

$$\begin{cases} W(h, P) = 0, \\ \dfrac{\partial W}{\partial h}\dot{h} + \dfrac{\partial W}{\partial P}\dot{P} = 0. \end{cases} \tag{A.11}$$

Substituting (6.10) into (A.11), and noting that $\dot{P}$ is linear with respect to Q, we obtain

$$\begin{cases} W(h, P) = 0, \\ A(h, P)\,Q + B(h, P) = 0, \end{cases} \tag{A.12}$$

where A and B are polynomials in h and P. By the same method as in Appendix 2, we find that the equations

$$\begin{cases} W(h, P) = 0, \\ A(h, P) = 0 \end{cases}$$

TABLE II

Some Values of σ_2 and τ

P	h	σ_2	τ
$\frac{8}{\sqrt{3\pi}}$	-0.122216	3.08033	3.22067
1.7	-0.367543	3.41741	4.43324
1.9	-0.459447	3.70997	5.83371
2	-0.48797	3.85487	6.6913

have only one solution (h^*, P^*) in the region (A.8), and $B(h^*, P^*) = 0$. Hence the set τ can be expressed as

$$\begin{cases} h = H(P), \\ Q = -\dfrac{B(H(P), P)}{A(H(P), P)}, \end{cases} \tag{A.13}$$

where $h = H(P)$ is the equation of σ_2' (see Appendix 1). For $P = P^*$, the second equation of (A.13) can be defined by taking the limit. Because of the smoothness of the vector field (6.10), τ must be a continuous curve on S_2 (i.e. the right hand side of the second equation of (A.13) is continuous at $P = P^*$). We compare some values of τ and σ_2 on S_2 in Table II, which shows that τ is above σ_2 near $P = 8/\sqrt{3\pi}$ and $P = 2$ on S_2. To finish the proof we perform two more steps:

(i) show that $\tau \cap \sigma_2 = \varnothing$.

(ii) check the direction of $\mathbf{v}$ with respect S_2 below σ_2 at any point. Note that $\partial W/\partial h$ and $\partial W/\partial P$ change their signs when P passes through P^*, it is better to check this fact at two points which are located on each side of $P^* \doteq 1.56633$ on S_2.

Step (i) We need to solve the equations

$$\begin{cases} W(h, P) = 0, \\ A(h, P)\,Q + B(h, P) = 0, \\ U(h, P, Q) = 0, \end{cases}$$

or equivalently (see Appendix 1)

$$\begin{cases} W(h, P) = 0, \\ A(h, P)\,Q + B(h, P) = 0, \\ V(h, P, Q) = 0. \end{cases} \tag{A.14}$$

TABLE III

$\mathbf{v} \cdot \mathbf{n}$ on S_2

P	h	$\dfrac{\partial U}{\partial h}$	$\dfrac{\partial U}{\partial P}$	Q	$\mathbf{v} \cdot \mathbf{n}$	Remark
$\dfrac{8}{\sqrt{3\pi}}$	-0.122216	+	+	3.05	+	Below σ_2
				3.1	+	Between σ_2 and τ
				3.8	–	Above τ
1.7	-0.367543	–	–	3.1	–	Below σ_2
				3.8	–	Between σ_2 and τ
				5	+	Above τ

By using the same procedure as in Appendix 2, we reduce (A.14) to the form

$$\begin{cases} A(h, P)\,Q + B(h, P) = 0, \\ W(h, P) = 0, \\ Y(h) = 0, \end{cases} \tag{A.15}$$

where $Y(h)$ is a polynomial of h. We find that (A.15) has a unique solution (h^*, P^*, Q^*) for (h, P) in the region (A.8), where $Q^* = -(B'_h H'(P) + B'_P)(A'_h H'(P) + A'_P)^{-1}\,|_{P^*} \doteq 3.67755$. But solving $V(h^*, P^*, Q) = 0$ we find $\hat{Q} = 3.22081$, hence (h^*, P^*, Q^*) is not a solution of (A.14). (As we mentioned in Appendix 2, by reducing (A.14) to (A.15), we multiply the equations by some polynomials, causing the extra solution.)

Step (ii) We list the result in Table III.

The result obtained in two steps, can be illustrated in Fig. 17.

We note that at (h^*, P^*), we could not use $(\partial W/\partial h, \partial W/\partial P)$. Since

$$H'(P^*) = -1.08753 = \left(\frac{\dot{h}}{\dot{P}}\right)\Bigg|_{(h^*,\, P^*,\, Q^*)},$$

and

$$\frac{\partial}{\partial Q}\left(\frac{\dot{h}}{\dot{P}}\right)\Bigg|_{(h^*,\, P^*)} = \frac{\dot{h}}{\dot{P}^2}\left(h^* + \frac{2}{3}P^*\right) > 0,$$

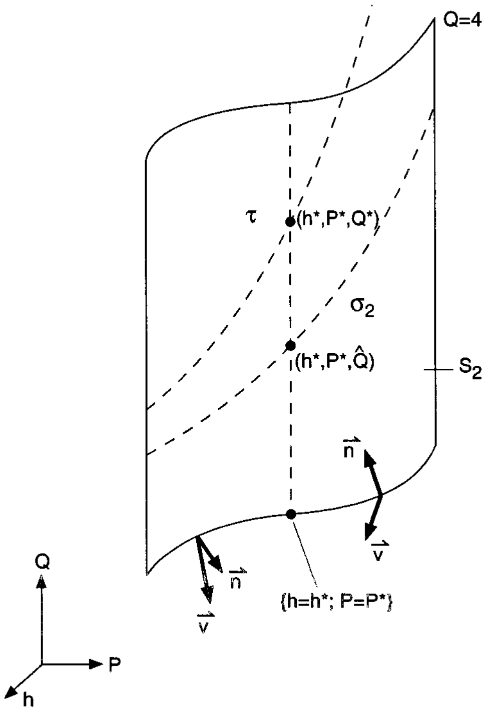

FIG. 17. Study of $\mathbf{v} \cdot \mathbf{n}$ along S_2.

we conclude that along the line $\{h = h^*,\ P = P^*\}$ and below the point (h^*, P^*, Q^*) (see Fig. 17) the vector field (6.10) is also transverse to S_2 and pointing towards $\{h = 0\}$.

ACKNOWLEDGMENTS

A part of this work and the proposed bifurcation diagram were presented in lectures at the University of Montreal (Dec. 1991), the University of Warsaw (March 1992), and the Universitat Autonomade Barcelona (April, 1992). The authors thank C. Rousseau for stimulating discussions and valuable comments and H. Żołądek and J. Llibre for interest in the paper. C. L. thanks the Limburgs Universitair Centrum for the hospitality and financial support during the academic year 1995–1996, when this work was completed.

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